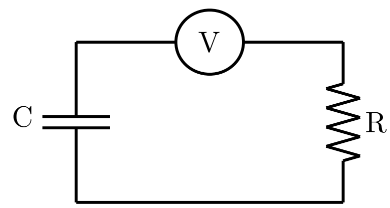
The RC Circuit

The RC circuit is the electrical circuit consisting of a resistor of resistance R, a capacitor of capacitance C and a voltage source arranged in series. If the charge on the capacitor is Q and the



current flowing in the circuit is I, the voltage across R and C are RI and $\frac{Q}{C}$ respectively. By the Kirchhoff's law that says that the voltage between any two points has to be independent of the path used to travel between the two points,

$$RI(t) + \tfrac{1}{C}Q(t) = V(t)$$

Assuming that R, C and V are known, this is still one differential equation in two unknowns, I and Q. However the two unknowns are related by $I(t) = \frac{dQ}{dt}(t)$ so that

$$RQ'(t) + \tfrac{1}{C}Q(t) = V(t)$$

The $V = 0$ Solution

If the applied voltage $V = 0$, this equation is separable and consequently easily solved.

$$R\tfrac{dQ}{dt} = -\tfrac{1}{C}Q \overset{\text{if } Q\neq 0}{\iff} \tfrac{dQ}{Q} = -\tfrac{1}{RC}dt \iff \int \tfrac{dQ}{Q} = -\tfrac{1}{RC}\int dt \iff \ln|Q| = -\tfrac{1}{RC}t + k$$
$$\iff Q(t) = Ke^{-t/RC}$$

where $K = \pm e^k$. At $t = 0$, $Q(0) = Ke^0 = K$, so $Q(t) = Q(0)e^{-t/RC}$. So the capacitor just discharges exponentially through the resistor.

The General Solution

When V is nonzero, theequation is no longer separable. But there is another trick that allows us to solve $RQ'(t) + \frac{1}{C}Q(t) = V(t)$ easily, by manipulating the left hand side into a perfect derivative. This trick works for any first order, linear, differential equation. "First order" means that the order of the highest order derivative acting on the unknown is one. "Linear" means that every term in the equation either is independent of the unknown function or is of the form $f(t)\frac{d^n}{dt^n}Q(t)$ for some known function $f(t)$ and some integer $n \geq 0$. First divide the equation by R, so that the coefficient of $Q'(t)$ is exactly one.

$$Q'(t) + \tfrac{1}{RC}Q(t) = \tfrac{1}{R}V(t) \tag{1}$$

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Now we are going to multiply the whole equation by a function $\mu(t)$ (called an integrating factor)

$$\mu(t)Q'(t) + \tfrac{1}{RC}\mu(t)Q(t) = \tfrac{1}{R}\mu(t)V(t)$$

This function will be carefully chosen so that the new left hand side is a perfect derivative. By the product rule $\frac{d}{dt}\big(\mu(t)Q(t)\big) = \mu(t)Q'(t) + \mu'(t)Q(t)$. So the new left hand side is exactly $\frac{d}{dt}\big(\mu(t)Q(t)\big)$ provided $\mu(t)$ obeys $\mu'(t) = \frac{1}{RC}\mu(t)$.

This equation for $\mu(t)$ is separable and so may be solved by the same technique that we used to solve $Q' = -\frac{1}{RC}Q$. But it is easier to just guess a solution. We are looking for a solution of $\mu'(t) = \frac{1}{RC}\mu(t)$. That is, differentiating $\mu(t)$ has to bring out a factor $\frac{1}{RC}$. So $\mu(t) = e^{t/RC}$ does the job. We only need one function $\mu(t)$, so choose $\mu(t) = e^{t/RC}$.

Multiplying (1) by $e^{t/RC}$ gives

$$\begin{aligned}
& e^{t/RC}Q'(t) + \tfrac{1}{RC}e^{t/RC}Q(t) = \tfrac{1}{R}e^{t/RC}V(t) \\
\Rightarrow\quad & \tfrac{d}{dt}\big(e^{t/RC}Q(t)\big) = \tfrac{1}{R}e^{t/RC}V(t) \\
\Rightarrow\quad & e^{t/RC}Q(t) = \tfrac{1}{R}\int e^{t/RC}V(t)\ dt \\
\Rightarrow\quad & \boxed{Q(t) = \tfrac{1}{R}e^{-t/RC}\textstyle\int e^{t/RC}V(t)\ dt}
\end{aligned}$$

For example, if $V(t) = V_0$, a constant, (that is, if a DC voltage is applied) $\int e^{t/RC}V(t)\ dt = \int e^{t/RC}V_0\ dt = RCV_0e^{t/RC} + k$ and

$$Q(t) = CV_0 + Ke^{-t/RC}$$

where $K = k/R$ is an arbitrary constant. In this case, the charge on the capacitor approachs CV_0, exponentially.

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