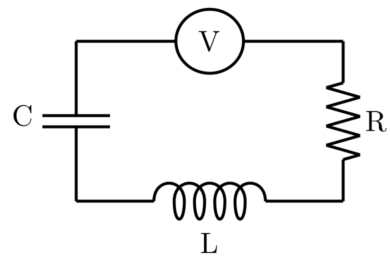
The RLC Circuit

The RLC circuit is the electrical circuit consisting of a resistor of resistance R, a coil of inductance L, a capacitor of capacitance C and a voltage source arranged in series. If the charge



on the capacitor is Q and the current flowing in the circuit is I, the voltage across R, L and C are RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$ respectively. By the Kirchhoff's law that says that the voltage between any two points has to be independent of the path used to travel between the two points,

$$LI'(t) + RI(t) + \tfrac{1}{C}Q(t) = V(t)$$

Assuming that R, L, C and V are known, this is still one differential equation in two unknowns, I and Q. However the two unknowns are related by $I(t) = \frac{dQ}{dt}(t)$ so that

$$LQ''(t) + RQ'(t) + \tfrac{1}{C}Q(t) = V(t)$$

or, differentiating with respect to t and then subbing in $\frac{dQ}{dt}(t) = I(t)$,

$$LI''(t) + RI'(t) + \tfrac{1}{C}I(t) = V'(t)$$

For an ac voltage source, choosing the origin of time so that $V(0) = 0$, $V(t) = E_0\sin(\omega t)$ and the differential equation becomes

$$LI''(t) + RI'(t) + \tfrac{1}{C}I(t) = \omega E_0\cos(\omega t) \tag{1}$$

The General Solution

We first guess one solution of (1) by trying $I_p(t) = A\sin(\omega t - \varphi)$ with the amplitude A and phase φ to be determined. That is, we are guessing that the circuit responds to an oscillating applied voltage with a current that oscillates with the same rate. For $I_p(t)$ to be a solution, we need

$$\begin{aligned} LI_p''(t) + RI_p'(t) + \tfrac{1}{C}I_p(t) &= \omega E_0\cos(\omega t) \qquad (1_p)\\ -L\omega^2 A\sin(\omega t-\varphi) + R\omega A\cos(\omega t-\varphi) + \tfrac{1}{C}A\sin(\omega t-\varphi) &= \omega E_0\cos(\omega t)\\ &= \omega E_0\cos(\omega t - \varphi + \varphi) \end{aligned}$$

and hence

$$\left(\tfrac{1}{C} - L\omega^2\right)A\sin(\omega t-\varphi) + R\omega A\cos(\omega t-\varphi) = \omega E_0\cos(\varphi)\cos(\omega t-\varphi) - \omega E_0\sin(\varphi)\sin(\omega t-\varphi)$$

Matching coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on the left and right hand sides gives

$$\big(L\omega^2 - \tfrac{1}{C}\big)A = \omega E_0 \sin(\varphi) \tag{2}$$

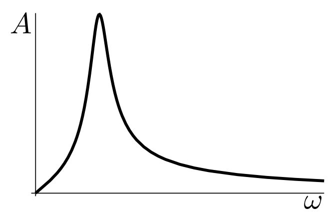
$$R\omega A = \omega E_0 \cos(\varphi) \tag{3}$$

It is now easy to solve for A and φ

$$\frac{(2)}{(3)} \Longrightarrow \tan(\varphi) = \frac{L\omega^2 - \frac{1}{C}}{R\omega} \qquad\qquad \Longrightarrow \varphi = \tan^{-1}\Big(\frac{L\omega}{R} - \frac{1}{RC\omega}\Big)$$

$$\sqrt{(2)^2 + (3)^2} \Longrightarrow \sqrt{\big(L\omega^2 - \tfrac{1}{C}\big)^2 + R^2\omega^2}\ A = \omega E_0 \Longrightarrow A = \frac{\omega E_0}{\sqrt{\big(L\omega^2 - \frac{1}{C}\big)^2 + R^2\omega^2}} \tag{4}$$

Naturally, different input frequencies ω give different output amplitudes A. Here is a graph of A against ω, with all other parameters held fixed.



Note that there is a small range of frequencies that give a large amplitude response. This is the phenomenon of resonance. It has been dramatically illustrated in, for example, the collapse of the Tacoma narrows bridge.

Now back to finding the general solution. Note that subtracting (1_p) from (1) gives

$$L(I - I_p)''(t) + R(I - I_p)'(t) + \tfrac{1}{C}(I - I_p)(t) = 0$$

That is, any solution of (1) differs from $I_p(t)$ by a solution of

$$LI''(t) + RI'(t) + \tfrac{1}{C}I(t) = 0 \tag{1_c}$$

This is called the complementary homogeneous equation for (1). We now guess many solutions to (1_c) by trying $I(t) = e^{rt}$, with the constant r to the determined. This guess is a solution of (1_c) if and only if

$$Lr^2e^{rt} + Rre^{rt} + \tfrac{1}{C}e^{rt} = 0 \iff Lr^2 + Rr + \tfrac{1}{C} = 0 \iff r = \tfrac{-R\pm\sqrt{R^2-4L/C}}{2L} \equiv r_{1,2} \tag{5}$$

We now know that e^{r_1t} and e^{r_2t} both obey (1_c). Because (1_c) is linear and homogeneous, this forces $c_1e^{r_1t} + c_2e^{r_2t}$ to also be a solution, for any values of the constants c_1 and c_2. (To check this, just sub $c_1e^{r_1t} + c_2e^{r_2t}$ into (1_c).) Assuming that $R^2 \neq 4L/C$, r_1 and r_2 are different and the general solution to (1_c) is $c_1e^{r_1t} + c_2e^{r_2t}$. (It is reasonable to guess that, to solve a differential equation involving a second derivative, one has to integrate twice so that the general solution contains two arbitrary constants.) Then, the general solution of (1) is

$$I(t) \ = \ c_1e^{r_1t} + c_2e^{r_2t} + A\sin(\omega t - \varphi)$$

with $r_1,\ r_2$ given in (5) and $A,\ \varphi$ given in (4). The arbitrary constants c_1 and c_2 are determined by initial conditions. However, when e^{r_1t} and e^{r_2t} damp out quickly, as is often the case, their values are not very important.