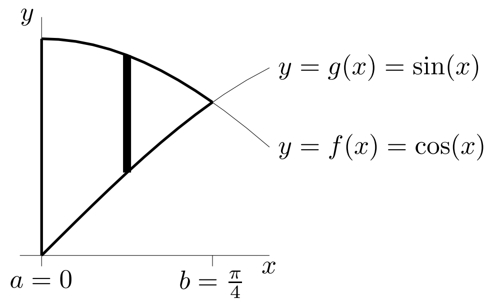
Centroid Example

Find the centroid of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{4}$.

Solution. We apply the formulae that the coordinates of the centroid (=centre of mass assuming constant density) of the region with top $y = f(x)$, bottom $y = g(x)$, left hand side $x = a$ and right hand side $x = b$ are

$$\bar{x} = \frac{\int_a^b x\,[f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}\,[f(x)^2 - g(x)^2]\,dx}{\int_a^b [f(x) - g(x)]\,dx}$$

Before we apply these formulae, we recall where they came from. Assume that the region has density one. Consider a thin vertical slice, of width dx, running from $(x, g(x))$ to $(x, f(x))$. It has area, and hence mass,



$[f(x)-g(x)]\,dx$. On this slice x is essentially constant. So the formula for $\bar{x}$ is just the formula for the (weighted) average of x over the whole region. On the slice y runs from $g(x)$ to $f(x)$. The average value of y on the slice is $\frac{1}{2}[f(x) + g(x)]$. Because $\frac{1}{2}[f(x) + g(x)][f(x) - g(x)] = \frac{1}{2}[f(x)^2 - g(x)^2]$, the formula for $\bar{y}$ is the formula for the average of y over the whole region.

In the given problem, $a = 0$, $b = \frac{\pi}{4}$, $f(x) = \cos x$ and $g(x) = \sin x$. Subbing these in to the denominator of the formulae for $\bar{x}$ and $\bar{y}$ gives

$$\begin{aligned}\int_a^b [f(x) - g(x)]\,dx &= \int_0^{\pi/4} [\cos(x) - \sin(x)]\,dx = \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4} \\ &= \big[\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\big] - \big[0 + 1\big] = \tfrac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1\end{aligned}$$

Subbing into the numerator of the formula for $\bar{x}$ gives

$$\int_a^b x\,[f(x) - g(x)]\,dx = \int_0^{\pi/4} x[\cos(x) - \sin(x)]\,dx$$

To integrate this, use integration by parts with $u = x$ and $dv = [\cos x - \sin x]\,dx$. So $du = dx$, $v = \sin x + \cos x$ and

$$\begin{aligned}\int_0^{\pi/4} x[\cos(x) - \sin(x)]\,dx &= x[\sin x + \cos x]\Big|_0^{\pi/4} - \int_0^{\pi/4} [\sin(x) + \cos(x)]\,dx \\ &= \tfrac{\pi}{4}\big[\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\big] - \Big[-\cos(x) + \sin(x)\Big]_0^{\pi/4} \\ &= \tfrac{2\pi}{4\sqrt{2}} - \Big[\big(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\big) - (-1 + 0)\Big] = \tfrac{\pi}{2\sqrt{2}} - 1\end{aligned}$$

Finally, subbing into the numerator of the formula for $\bar{y}$ gives

$$\int_a^b \tfrac{1}{2}\,[f(x)^2 - g(x)^2]\,dx = \int_0^{\pi/4} \tfrac{1}{2}\,[\cos^2 x - \sin^2 x]\,dx = \int_0^{\pi/4} \tfrac{1}{2}\,\cos(2x)\,dx = \Big[\tfrac{1}{4}\sin(2x)\Big]_0^{\pi/4} = \tfrac{1}{4}$$

Putting the formulae together

$$\bar{x} = \frac{\int_a^b x\,[f(x) - g(x)]\,dx}{\int_a^b [f(x) - g(x)]\,dx} = \frac{\frac{\pi}{2\sqrt{2}} - 1}{\sqrt{2} - 1} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}\,[f(x)^2 - g(x)^2]\,dx}{\int_a^b [f(x) - g(x)]\,dx} = \frac{\frac{1}{4}}{\sqrt{2} - 1}$$