The Partial Fractions Decomposition

The Simplest Case

In the most common partial fraction decomposition, we split up

$$\tfrac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

into a sum of the form

$$\tfrac{A_1}{x-a_1}+\cdots+\tfrac{A_d}{x-a_d}$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $d-1$. To do so, we shall repeatedly apply the following Lemma. (The word Lemma just signifies that the result is not that important – it is only used as a tool to prove a more important result.)

Lemma 1 *Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n \le d$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p<d$ and a numbers A such that*

$$\tfrac{N(x)}{D(x)\,(x-a)}=\tfrac{P(x)}{D(x)}+\tfrac{A}{x-a}$$

Proof: To save writing, let $z=x-a$. Then $\tilde N(z)=N(z+a)$ and $\tilde D(z)=D(z+a)$ are again polynomials of degree n and d respectively, $\tilde D(0)=D(a)\neq 0$ and we have to find a polynomial $\tilde P(z)$ of degree $p<d$ and a number A such that

$$\tfrac{\tilde N(z)}{\tilde D(z)\,z}=\tfrac{\tilde P(z)}{\tilde D(z)}+\tfrac{A}{z}=\tfrac{\tilde P(z)z+A\tilde D(z)}{\tilde D(z)\,z}$$

or equivalently, such that

$$\tilde P(z)z+A\tilde D(z)=\tilde N(z)$$

Now look at the polynomial on the left hand side. Every term in $\tilde P(z)z$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A\tilde D(z)$, which is $A\tilde D(0)$. The constant term on the right hand side is $\tilde N(0)$. So the constant terms on the left and right hand sides are the same if we choose $A=\frac{\tilde N(0)}{\tilde D(0)}$. Recall that $\tilde D(0)$ cannot be zero. Now move $A\tilde D(z)$ to the right hand side.

$$\tilde P(z)z=\tilde N(z)-A\tilde D(z)$$

The constant terms in $\tilde{N}(z)$ and $A\tilde{D}(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$. Since $\tilde{N}(z)$ is of degree at most d and $A\tilde{D}(z)$ is of degree exactly d, $\tilde{N}_1$ is a polynomial of degree $d-1$. It now suffices to choose $\tilde{P}(z) = \tilde{N}_1(z)$. ■

Now back to

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

Apply Lemma 1, with $D(x) = (x-a_2)\times\cdots\times(x-a_d)$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)}$$

for some polynomial P of degree at most $d-2$ and some number A_1. Apply Lemma 1 a second time, with $D(x) = (x-a_3)\times\cdots\times(x-a_d)$, $N(x) = P(x)$ and $a = a_2$. It says

$$\frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)} = \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

for some polynomial Q of degree at most $d-3$ and some number A_2. At this stage, we know that

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

If we just keep going, repeatedly applying Lemma 1, we eventually end up with

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_d}{x-a_d}$$

The general case with linear factors

Now consider splitting

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

into a sum of the form

$$\left[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\right]$$

Note that, if we allow ourselves to use complex roots, this is the general case. We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $n_1 + \cdots + n_d - 1$. To do so, we shall repeatedly apply the following Lemma.

Lemma 2 *Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n < d+m$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that*

$$\frac{N(x)}{D(x)\,(x-a)^m} = \frac{P(x)}{D(x)} + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$$

Proof: To save writing, let $z = x - a$. Then $\tilde N(z) = N(z+a)$ and $\tilde D(z) = D(z+a)$ are polynomials of degree n and d respectively, $\tilde D(0) = D(a) \neq 0$ and we have to find a polynomial $\tilde P(z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\begin{aligned}\frac{\tilde N(z)}{\tilde D(z)\,z^m} &= \frac{\tilde P(z)}{\tilde D(z)} + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_m}{z^m}\\ &= \frac{\tilde P(z)z^m + A_1 z^{m-1}\tilde D(z) + A_2 z^{m-2}\tilde D(z) + \cdots + A_m \tilde D(z)}{\tilde D(z)\,z^m}\end{aligned}$$

or equivalently, such that

$$\tilde P(z)z^m + A_1 z^{m-1}\tilde D(z) + A_2 z^{m-2}\tilde D(z) + \cdots + A_{m-1} z\tilde D(z) + A_m \tilde D(z) = \tilde N(z)$$

Now look at the polynomial on the left hand side. Every single term on the left hand side, except for the very last one, $A_m\tilde D(z)$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A_m\tilde D(z)$, which is $A_m\tilde D(0)$. The constant term on the right hand side is $\tilde N(0)$. So the constant terms on the left and right hand sides are the same if we choose $A_m = \frac{\tilde N(0)}{\tilde D(0)}$. Recall that $\tilde D(0) \neq 0$. Now move $A_m\tilde D(z)$ to the right hand side.

$$\tilde P(z)z^m + A_1 z^{m-1}\tilde D(z) + A_2 z^{m-2}\tilde D(z) + \cdots + A_{m-1} z\tilde D(z) = \tilde N(z) - A_m \tilde D(z)$$

The constant terms in $\tilde N(z)$ and $A_m\tilde D(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde N_1(z)z$ with $\tilde N_1$ a polynomial of degree at most $d+m-2$. (Recall that $\tilde N$ is of degree at most $d+m-1$ and $\tilde D$ is of degree at most d.) Divide the whole equation by z.

$$\tilde P(z)z^{m-1} + A_1 z^{m-2}\tilde D(z) + A_2 z^{m-3}\tilde D(z) + \cdots + A_{m-1}\tilde D(z) = \tilde N_1(z)$$

Now, we can repeat the previous argument. The constant term on the left hand side, which is exactly $A_{m-1}\tilde D(0)$ matchs the constant term on the right hand side, which is $\tilde N_1(0)$ if we choose $A_{m-1} = \frac{\tilde N_1(0)}{\tilde D(0)}$. With this choice of A_{m-1}

$$\tilde P(z)z^{m-1} + A_1 z^{m-2}\tilde D(z) + A_2 z^{m-3}\tilde D(z) + \cdots + A_{m-2} z\tilde D(z) = \tilde N_1(z) - A_{m-1}\tilde D(z) = \tilde N_2(z)z$$

with $\tilde{N}_2$ a polynomial of degree at most $d+m-3$. Divide by z and continue. After m steps like this, we end up with

$$\tilde{P}(z)z = \tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$$

after having chosen $A_1 = \frac{\tilde{N}_{m-1}(0)}{\tilde{D}(0)}$. There is no constant term on the right side so that $\tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$ is of the form $\tilde{N}_m(z)z$ with $\tilde{N}_m$ a polynomial of degree $d-1$. Choosing $\tilde{P}(z) = \tilde{N}_m(z)$ completes the proof. ■

Now back to

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

Apply Lemma 2, with $D(x) = (x-a_2)^{n_2}\times\cdots\times(x-a_d)^{n_d}$, $m = n_1$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x-a)^{n_1}} + \frac{P(x)}{(x-a_2)^{n_2}\times\cdots\times(x-a_d)^{n_d}}$$

Apply Lemma 2 a second time, with $D(x) = (x-a_3)^{n_3}\times\cdots\times(x-a_d)^{n_d}$, $N(x) = P(x)$, $m = n_2$ and $a = a_2$. And so on. Eventually, we end up with

$$\left[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\right]$$