The Form of Partial Fractions Decompositions

In the following it is assumed that

- N and D are polynomials with the degree of N strictly smaller than the degree of D.
- $K$is a constant.
- a_1, a_2, $\cdots$, a_j are all different.
- m_1, m_2, $\cdots$, m_j, n_1, n_2, $\cdots$, n_k are all strictly positive integers.
- $x^2+b_1x+c_1$, $x^2+b_2x+c_2$, $\cdots$, $x^2+b_kx+c_k$ are all different.

Simple Linear Factor Case

If $D(x)=K(x-a_1)(x-a_2)\cdots(x-a_j)$ then

$$\frac{N(x)}{D(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{A_j}{x-a_j}$$

The proof that such a decomposition always exists is given in notes that I have posted on our course home page.

General Linear Factor Case

If $D(x)=K(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}$ then

$$\begin{aligned}\frac{N(x)}{D(x)}=&\frac{B_{1,m_1-1}x^{m_1-1}+\cdots+B_{1,1}x+B_{1,0}}{(x-a_1)^{m_1}}\\&+\frac{B_{2,m_2-1}x^{m_2-1}+\cdots+B_{2,1}x+B_{2,0}}{(x-a_2)^{m_2}}+\cdots\\&+\frac{B_{j,m_j-1}x^{m_j-1}+\cdots+B_{j,1}x+B_{j,0}}{(x-a_j)^{m_j}}\end{aligned}$$

Each numerator is the most general polynomial with degree strictly smaller than that of the denominator. The proof that such a decomposition always exists is given in notes that I have posted on our course home page.

It is generally mechanically advantageous to rewrite

$$\tfrac{B_{m-1}x^{m-1}+\cdots+B_1x+B_0}{(x-a)^m} = \tfrac{A_1}{x-a} + \tfrac{A_2}{(x-a)^2} + \cdots + \tfrac{A_m}{(x-a)^m}$$

To see that these forms are equivalent, rename $x - a = y$. Then

$$\begin{aligned} \tfrac{B_{m-1}x^{m-1}+\cdots+B_1x+B_0}{(x-a)^m} &= \tfrac{B_{m-1}(y+a)^{m-1}+\cdots+B_1(y+a)+B_0}{y^m} \\ &= \tfrac{A_1y^{m-1}+\cdots+A_{m-1}y+A_m}{y^m} \\ &= \tfrac{A_1}{x-a} + \cdots + \tfrac{A_{m-1}}{(x-a)^{m-1}} + \tfrac{A_m}{(x-a)^m} \end{aligned}$$

The rewritten decomposition is

$$\begin{aligned} \frac{N(x)}{D(x)} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} \\ + \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}} + \cdots \\ + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \end{aligned}$$

Simple Linear and Quadratic Factor Case

If $D(x) = K(x-a_1)\cdots(x-a_j)(x^2+b_1x+c_1)\cdots(x^2+b_kx+c_k)$ then

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_j}{x-a_j} + \frac{B_1x+C_1}{x^2+b_1x+c_1} + \cdots + \frac{B_kx+C_k}{x^2+b_kx+c_k}$$

General Linear and Quadratic Factor Case

If $D(x) = K(x-a_1)^{m_1}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$ then

$$\begin{aligned}\frac{N(x)}{D(x)} = {} & \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \cdots \\ & + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \\ & + \frac{B_{1,1}x+C_{1,1}}{x^2+b_1x+c_1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,n_1}x+C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}} + \cdots \\ & + \frac{B_{k,1}x+C_{k,1}}{x^2+b_kx+c_k} + \frac{B_{k,2}x+C_{k,2}}{(x^2+b_kx+c_k)^2} + \cdots + \frac{B_{k,n_k}x+C_{1,n_k}}{(x^2+b_kx+c_k)^{n_k}}\end{aligned}$$