Partial Fractions Examples

Partial fractions is the name given to a technique of integration that may be used to integrate any ratio of polynomials. A ratio of polynomials is called a rational function. Suppose that $N(x)$ and $D(x)$ are polynomials. The basic strategy is to write $\frac{N(x)}{D(x)}$ as the sum of a polynomial $P(x)$ (this term is present only if the degree of $N(x)$ is strictly bigger than the degree of $D(x)$) and a number of rational functions of the particularly simple forms $\frac{A}{(ax+b)^n}$ and $\frac{Ax+B}{(ax^2+bx+c)^m}$.

Example 1: $\dfrac{N(x)}{D(x)} = \dfrac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}$

Step 1. The first step is applied only when the degree of the numerator $N(x)$ is at least as large as the degree of the denominator $D(x)$. In this example, the numerator has degree four and the denominator has degree three. As $4 \geq 3$, we must execute the first step, which is to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. This step is accomplished by long division – the same long division you learned in public school with the base 10 replaced by x. We start by observing that to get from the highest degree term in the denominator (x^3) to the highest degree term in the numerator (x^4), we have to multiply by x. So we write,

$$\begin{array}{r|l} & x \\ \cline{2-2} x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \end{array}$$

Now we subtract x times the denominator x^3+3x^2+7x+5, which is $x^4+3x^3+7x^2+5x$, from the numerator.

$$\begin{array}{r|l} & x \\ \cline{2-2} x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\ & x^4+3x^3+\ 7x^2+\ 5x \qquad \longleftarrow x(x^3+3x^2+7x+5) \\ \cline{2-2} & \qquad 2x^3+\ 9x^2+21x+22 \end{array}$$

The remainder was $2x^3+9x^2+21x+22$. To get from the highest degree term in the denominator (x^3) to the highest degree term in the remainder ($2x^3$), we have to multiply by 2. So we write,

$$\begin{array}{r|l} & x+\ \ 2 \\ \cline{2-2} x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\ & x^4+3x^3+\ 7x^2+\ 5x \\ \cline{2-2} & \qquad 2x^3+\ 9x^2+21x+22 \end{array}$$

Now we subtract 2 times the denominator x^3+3x^2+7x+5, which is $2x^3+6x^2+14x+10$, from the remainder.

$$
\begin{array}{r|l}
 & \quad x + \quad 2 \\
\cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\
 & x^4+3x^3+\ \ 7x^2+\ \ 5x \qquad \longleftarrow\ x(x^3+3x^2+7x+5) \\
\cline{2-2}
 & \qquad 2x^3+\ \ 9x^2+21x+22 \\
 & \qquad 2x^3+\ \ 6x^2+14x+10 \qquad \longleftarrow\ 2(x^3+3x^2+7x+5) \\
\cline{2-2}
 & \qquad\qquad\quad 3x^2+\ \ 7x+12
\end{array}
$$

This leaves a remainder of $3x^2 + 7x + 12$. Because the remainder has degree 2, which is smaller than the degree of the denominator, which is 3, we stop.

In this example, when we subtracted $x(x^3 + 3x^2 + 7x + 5)$ and $2(x^3 + 3x^2 + 7x + 5)$ from $x^4 + 5x^3 + 16x^2 + 26x + 22$ we ended up with $3x^2 + 7x + 12$. That is,

$$x^4 + 5x^3 + 16x^2 + 26x + 22 - x(x^3 + 3x^2 + 7x + 5) - 2(x^3 + 3x^2 + 7x + 5) = 3x^2 + 7x + 12$$

or, collecting the two terms proportional to $(x^3 + 3x^2 + 7x + 5)$

$$x^4 + 5x^3 + 16x^2 + 26x + 22 - (x + 2)(x^3 + 3x^2 + 7x + 5) = 3x^2 + 7x + 12$$

Moving the $(x+2)(x^3+3x^2+7x+5)$ to the right hand side and dividing the whole equation by $x^3 + 3x^2 + 7x + 5$ gives

$$\boxed{\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{3x^2 + 7x + 12}{x^3 + 3x^2 + 7x + 5}}$$

This is of the form $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation.

$$
\begin{array}{r|l}
 & \quad x + \quad 2 \ \longleftarrow\!\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\ P(x) \\
\cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\
 & x^4+3x^3+\ \ 7x^2+\ \ 5x \\
\cline{2-2}
 & \qquad 2x^3+\ \ 9x^2+21x+22 \\
 & \qquad 2x^3+\ \ 6x^2+14x+10 \\
\cline{2-2}
 & \qquad\qquad\quad 3x^2+\ \ 7x+12 \qquad \longleftarrow\ R(x)
\end{array}
$$

Step 2. The second step is to factor the denominator $D(x) = x^3 + 3x^2 + 7x + 5$. Fortunately, there is a trick available to help us find this factorization. The trick exploits the fact that most polynomials that appear in homework assignments and on tests have integer coefficients and some integer roots. Any integer root of a polynomial that has integer coefficients, like $D(x) = x^3 + 3x^2 + 7x + 5$, must divide the constant term of the polynomial exactly. So any integer root of $x^3 + 3x^2 + 7x + 5$ must divide 5 exactly. The only integers which can be roots of $D(x)$ are ± 1 and ± 5. To test if $+1$ is a root, we sub it into $D(x)$:

$$D(1) = 1^3 + 3(1)^2 + 7(1) + 5 = 16$$

As $D(1) \neq 0$, 1 is not a root of $D(x)$. To test if -1 is a root, we sub it into $D(x)$:

$$D(-1) = (-1)^3 + 3(-1)^2 + 7(-1) + 5 = -1 + 3 - 7 + 5 = 0$$

As $D(-1) = 0$, -1 is a root of $D(x)$. As -1 is a root of $D(x)$, $\big(x - (-1)\big) = (x+1)$ must factor $D(x)$ exactly. We can factor the $(x+1)$ out of $D(x) = x^3 + 3x^2 + 7x + 5$ by long division once again.

$$
\begin{array}{r|rll}
 & x^2 + 2x + 5 & & \\
\hline
x+1 & x^3 + 3x^2 + 7x + 5 & & \\
 & x^3 + x^2 & \longleftarrow & x^2(x+1) \\
\hline
 & 2x^2 + 7x + 5 & & \\
 & 2x^2 + 2x & \longleftarrow & 2x(x+1) \\
\hline
 & 5x + 5 & & \\
 & 5x + 5 & \longleftarrow & 5(x+1) \\
\hline
 & 0 & &
\end{array}
$$

This time, when we subtracted $x^2(x+1)$ and $2x(x+1)$ and $5(x+1)$ from $x^3 + 3x^2 + 7x + 5$ we ended up with 0. Hence

$$x^3 + 3x^2 + 7x + 5 - x^2(x+1) - 2x(x+1) - 5(x+1) = 0$$

or

$$x^3 + 3x^2 + 7x + 5 = x^2(x+1) + 2x(x+1) + 5(x+1)$$

or

$$x^3 + 3x^2 + 7x + 5 = (x^2 + 2x + 5)(x+1)$$

We still have a quadratic factor, $x^2 + 2x + 5$. We can attempt to factor it by using the high school formula $(-b \pm \sqrt{b^2 - 4ac})/(2a)$ to find the roots of $x^2 + 2x + 5$. For $x^2 + 2x + 5$,

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

contains the square root of a negative number. So, without the use of complex numbers, $x^2 + 2x + 5$ cannot be factored. We have reached the end of step 2. At this point we have

$$\boxed{\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{3x^2 + 7x + 12}{(x+1)(x^2 + 2x + 5)}}$$

Step 3. The third step is to write $\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)}$ in the form

$$\frac{3x^2 + 7x + 12}{(x+1)(x^2 + 2x + 5)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 2x + 5}$$

for some constants A, B and C. To determine the values of the constants A, B, C, we put the right hand side back over the common denominator $(x+1)(x^2+2x+5)$.

$$\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+5} = \frac{A(x^2+2x+5)+(Bx+C)(x+1)}{(x+1)(x^2+2x+5)}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$3x^2+7x+12 = A(x^2+2x+5)+(Bx+C)(x+1)$$

There are a couple of different ways to determine the values of A, B and C from this equation.

The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

$$3x^2+7x+12 = (A+B)x^2+(2A+B+C)x+(5A+C)$$

For these two polynomials to be the same, the coefficient of x^2 on the left hand side and the coefficient of x^2 on the right hand side must be the same. Similarly the coefficients of x^1 must match and the coefficients of x^0 must match. This gives us a system of three equations

$$A+B=3 \qquad 2A+B+C=7 \qquad 5A+C=12$$

in the three unknowns A, B, C. We can solve this system by using the first equation, $A+B=3$, to determine B in terms of A, $B=3-A$. Substituting this into the remaining two equations eliminates the A's from these two equations, leaving two equations in the two unknowns B and C.

$$\begin{array}{rrrr} & A=3-B & 2A+B+C=7 & 5A+C=12 \\ \Rightarrow & & 2(3-B)+B+C=7 & 5(3-B)+C=12 \\ \Rightarrow & & -B+C=1 & -5B+C=-3 \end{array}$$

Now we can use the equation, $-B+C=1$, to determine B in terms of C, $B=C-1$. Substituting this into the remaining equation eliminates the B's leaving an equation in the one unknown C, which is easy to solve.

$$\begin{array}{rrr} & B=C-1 & -5B+C=-3 \\ \Rightarrow & & -5(C-1)+C=-3 \\ \Rightarrow & & -4C=-8 \end{array}$$

So $C=2$, $B=C-1=1$ and $A=3-B=2$.

There is a second, more efficient, procedure for determining A, B and C from

$$3x^2+7x+12 = A(x^2+2x+5)+(Bx+C)(x+1)$$

This equation must be true for all values of x. In particular, it must be true for $x=-1$. When $x=-1$, the factor $(x+1)$ multiplying $Bx+C$ is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$3x^2+7x+12\Big|_{x=-1} = A(x^2+2x+5)\Big|_{x=-1} + (Bx+C)(x+1)\Big|_{x=-1} \Longrightarrow 8=4A \Longrightarrow A=2$$

Sub this value of A back in and simplify.

$$\begin{aligned} 3x^2 + 7x + 12 &= 2(x^2 + 2x + 5) + (Bx + C)(x + 1) \\ x^2 + 3x + 2 &= (Bx + C)(x + 1) \end{aligned}$$

Since $(x + 1)$ is a factor on the right hand side, it must also be a factor on the left hand side.

$$(x + 2)(x + 1) = (Bx + C)(x + 1) \quad \Rightarrow \quad (x + 2) = (Bx + C) \quad \Rightarrow \quad B = 1,\ C = 2$$

Subbing these values of A, B and C into $\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+5}$ we have

$$\boxed{\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{2}{x + 1} + \frac{x + 2}{x^2 + 2x + 5}}$$

Step 4. The final step is to integrate. The first two pieces are easy.

$$\int (x + 2)\, dx = \tfrac{1}{2}x^2 + 2x \qquad \int \frac{2}{x + 1}\, dx = 2 \ln |x + 1|$$

(I'm leaving the arbitrary constant to the end of the computation.) The final piece is a little harder. The idea is to complete the square in the denominator

$$\frac{x + 2}{x^2 + 2x + 5} = \frac{x + 2}{(x + 1)^2 + 4}$$

and then make a change of variables to make the fraction look like $\frac{ay+b}{y^2+1}$. In this case

$$\frac{x + 2}{(x + 1)^2 + 4} = \frac{1}{4} \frac{x + 2}{(\frac{x+1}{2})^2 + 1}$$

so we make the change of variables $y = \frac{x+1}{2},\ dy = \frac{dx}{2},\ x = 2y - 1,\ dx = 2\, dy$

$$\begin{aligned} \int \frac{x + 2}{(x + 1)^2 + 4}\, dx &= \frac{1}{4} \int \frac{x + 2}{\left(\frac{x+1}{2}\right)^2 + 1}\, dx = \frac{1}{4} \int \frac{(2y - 1) + 2}{y^2 + 1} 2\, dy = \frac{1}{2} \int \frac{2y + 1}{y^2 + 1}\, dy \\ &= \int \frac{y}{y^2 + 1}\, dy + \frac{1}{2} \int \frac{1}{y^2 + 1}\, dy \end{aligned}$$

Both integrals are easily evaluated, using the substitution $u = y^2 + 1$, $du = 2y\, dy$ for the first.

$$\begin{aligned} \int \frac{y}{y^2 + 1}\, dy &= \int \frac{1}{u} \frac{du}{2} = \tfrac{1}{2} \ln |u| = \tfrac{1}{2} \ln(y^2 + 1) = \tfrac{1}{2} \ln \left[\left(\tfrac{x+1}{2}\right)^2 + 1 \right] \\ \frac{1}{2} \int \frac{1}{y^2 + 1}\, dy &= \tfrac{1}{2} \tan^{-1} y = \tfrac{1}{2} \tan^{-1} \left(\tfrac{x+1}{2}\right) \end{aligned}$$

That's finally it. Putting all of the pieces together

$$\boxed{\int \tfrac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}\, dx = \tfrac{1}{2}x^2 + 2x + 2 \ln |x + 1| + \tfrac{1}{2} \ln \left[\left(\tfrac{x+1}{2}\right)^2 + 1 \right] + \tfrac{1}{2} \tan^{-1} \left(\tfrac{x+1}{2}\right) + C}$$

Example 2: $\frac{N(x)}{D(x)} = \frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4}$

Step 1. The first step to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. By long division

$$\begin{array}{r|l} & x + \quad 4 \\ \hline x^3 + 5x^2 + 8x + 4 & x^4 + 9x^3 + 31x^2 + 49x + 27 \\ & x^4 + 5x^3 + \;\; 8x^2 + \;\; 4x \\ \hline & \qquad 4x^3 + 23x^2 + 45x + 27 \\ & \qquad 4x^3 + 20x^2 + 32x + 16 \\ \hline & \qquad\qquad\quad 3x^2 + 13x + 11 \end{array}$$

so

$$\frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4} = x + 4 + \frac{3x^2 + 13x + 11}{x^3 + 5x^2 + 8x + 4}$$

Step 2. The second step is to factorize $D(x) = x^3 + 5x^2 + 8x + 4$. Any integer root of $D(x)$ must divide the constant term, 4, exactly. Only ± 1, ± 2, ± 4 can be integer roots of $x^3 + 5x^2 + 8x + 4$. We test to see if ± 1 are roots.

$$\begin{aligned} D(1) &= (1)^3 + 5(1)^2 + 8(1) + 4 \neq 0 && \Rightarrow x = 1 \text{ is not a root} \\ D(-1) &= (-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0 && \Rightarrow x = -1 \text{ is a root} \end{aligned}$$

So $(x + 1)$ must divide $x^3 + 5x^2 + 8x + 4$. By long division

$$\begin{array}{r|l} & x^2 + \;\; 4x + \quad 4 \\ \hline x + 1 & x^3 + 5x^2 + 8x + 4 \\ & x^3 + \;\; x^2 \\ \hline & \qquad 4x^2 + 8x + 4 \\ & \qquad 4x^2 + 4x \\ \hline & \qquad\qquad\quad 4x + 4 \\ & \qquad\qquad\quad 4x + 4 \\ \hline & \qquad\qquad\qquad\quad 0 \end{array}$$

so

$$x^3 + 5x^2 + 8x + 4 = (x + 1)(x^2 + 4x + 4) = (x + 1)(x + 2)(x + 2)$$

This is the end of step 2. We now know

$$\frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4} = x + 4 + \frac{3x^2 + 13x + 11}{(x + 1)(x + 2)^2}$$

Step 3. The third step is to write $\frac{3x^2+13x+11}{(x+1)(x+2)^2}$ in the form

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

for some constants A, B and C. To determine the values of the constants A, B, C, we put the right hand side back over the common denominator $(x+1)(x+2)^2$.

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} = \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$3x^2 + 13x + 11 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

This must be true for all values of x. In particular, it must be true for $x = -1$. When $x = -1$, the factor $(x+1)$ multiplying B and C is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$3x^2 + 13x + 11\Big|_{x=-1} = A(x+2)^2\Big|_{x=-1} + B(x+1)(x+2)\Big|_{x=-1} + C(x+1)\Big|_{x=-1} \Longrightarrow 1 = A$$

Sub this value of A back in and simplify.

$$\begin{aligned} 3x^2 + 13x + 11 &= (1)(x+2)^2 + B(x+1)(x+2) + C(x+1) \\ 2x^2 + 9x + 7 &= B(x+1)(x+2) + C(x+1) = (xB + 2B + C)(x+1) \end{aligned}$$

Since $(x+1)$ is a factor on the right hand side, it must also be a factor on the left hand side.

$$(2x+7)(x+1) = (xB + 2B + C)(x+1) \quad \Rightarrow \quad (2x+7) = (xB + 2B + C)$$

For the coefficients of x to match, B must be 2. For the constant terms to match, $2B + C$ must be 7, so C must be 3. Subbing into $\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$, we now have

$$\frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4} = x + 4 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

Step 4. The final step is to integrate

$$\begin{aligned} \int \frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4}\,dx &= \int (x+4)\,dx + \int \frac{1}{x+1}\,dx + \int \frac{2}{x+2}\,dx + \int \frac{3}{(x+2)^2}\,dx \\ &= \boxed{\tfrac{1}{2}x^2 + 4x + \ln|x+1| + 2\ln|x+2| - \frac{3}{x+2} + C} \end{aligned}$$