

Integration of $\sec x$ and $\sec^3 x$

$\int \sec x \, dx$ – by trickery

The standard trick used to integrate $\sec x$ is to multiply the integrand by $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$ and then substitute $y = \sec x + \tan x$, $dy = (\sec x \tan x + \sec^2 x) \, dx$.

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{dy}{y} = \ln |y| + C \\ &= \ln |\sec x + \tan x| + C\end{aligned}$$

$\int \sec x \, dx$ – by partial fractions

Another method for integrating $\int \sec x \, dx$, that is more tedious, but less dependent on a memorized trick, is to convert $\int \sec x \, dx$ into the integral of a rational function using the substitution $y = \sin x$, $dy = \cos x \, dx$ and then use partial fractions.

$$\begin{aligned}\int \sec x \, dx &= \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx = \int \frac{1}{1 - y^2} \, dy \\ &= \int \frac{1}{2} \left[\frac{1}{1+y} + \frac{1}{1-y} \right] \, dy = \frac{1}{2} \int \left[\frac{1}{y+1} - \frac{1}{y-1} \right] \, dy = \frac{1}{2} \left[\ln |y+1| - \ln |y-1| \right] + C \\ &= \frac{1}{2} \ln \left| \frac{y+1}{y-1} \right| + C = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C\end{aligned}$$

To see that this answer is really the same as the one above, note that

$$\begin{aligned}\frac{\sin x + 1}{\sin x - 1} &= \frac{(\sin x + 1)^2}{\sin^2 x - 1} = \frac{(\sin x + 1)^2}{-\cos^2 x} \\ \Rightarrow \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| &= \frac{1}{2} \ln \left| \frac{(\sin x + 1)^2}{-\cos^2 x} \right| = \frac{1}{2} \ln \left| \frac{(\sin x + 1)^2}{\cos^2 x} \right| = \ln \left| \frac{\sin x + 1}{\cos x} \right| = \ln |\tan x + \sec x|\end{aligned}$$

$\int \sec^3 x \, dx$ – by trickery

The standard trick used to evaluate $\int \sec^3 x \, dx$ is integration by parts with $u = \sec x$, $dv = \sec^2 x \, dx$, $du = \sec x \tan x \, dx$, $v = \tan x$.

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx = \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\int \sec^3 x \, dx = \sec x \tan x - \int [\sec^3 x - \sec x] \, dx = \sec x \tan x + \ln |\sec x + \tan x| + C - \int \sec^3 x \, dx$$

where we used $\int \sec x \, dx = \ln |\sec x + \tan x| + C$. Now moving the $\int \sec^3 x \, dx$ from the right hand side to the left hand side

$$\begin{aligned}2 \int \sec^3 x \, dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\ \Rightarrow \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C\end{aligned}$$

for a new arbitrary constant C .

$\int \sec^3 x \, dx$ – by partial fractions

Another method for integrating $\int \sec^3 x \, dx$, that is more tedious, but less dependent on trickery, is to convert $\int \sec^3 x \, dx$ into the integral of a rational function using the substitution $y = \sin x$, $dy = \cos x \, dx$ and then use partial fractions.

$$\begin{aligned}\int \sec^3 x \, dx &= \int \frac{1}{\cos^3 x} \, dx = \int \frac{\cos x}{\cos^4 x} \, dx = \int \frac{\cos x}{[1 - \sin^2 x]^2} \, dx = \int \frac{1}{[1 - y^2]^2} \, dy = \int \frac{1}{[y^2 - 1]^2} \, dy \\ &= \int \left[\frac{1}{2} \left(\frac{1}{y-1} - \frac{1}{y+1} \right) \right]^2 \, dy = \frac{1}{4} \int \left[\frac{1}{(y-1)^2} - \frac{2}{(y-1)(y+1)} + \frac{1}{(y+1)^2} \right] \, dy \\ &= \frac{1}{4} \int \left[\frac{1}{(y-1)^2} - \frac{1}{y-1} + \frac{1}{y+1} + \frac{1}{(y+1)^2} \right] \, dy \\ &= \frac{1}{4} \left[-\frac{1}{y-1} - \ln |y-1| + \ln |y+1| - \frac{1}{y+1} \right] + C = -\frac{1}{4} \frac{2y}{y^2-1} + \frac{1}{4} \ln \left| \frac{y+1}{y-1} \right| + C \\ &= \frac{1}{2} \frac{y}{1-y^2} + \frac{1}{4} \ln \left| \frac{y+1}{y-1} \right| + C = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{4} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C\end{aligned}$$