Techniques of Integration – Substitution

The substitution rule for simplifying integrals is just the chain rule rewritten in terms of integrals. Suppose that $F(y)$ is a function whose derivative is $f(y)$. That is, $F(y)$ is an indefinite integral for $f(y)$ so that

$$\int f(y)\,dy = F(y) + C$$

Then the chain rule says that, for any function $y(x)$,

$$\tfrac{d}{dx}F\big(y(x)\big) = F'\big(y(x)\big)y'(x) = f\big(y(x)\big)y'(x)$$

So $F\big(y(x)\big)$ is one function with derivative $f\big(y(x)\big)y'(x)$ and $F\big(y(x)\big)$ is an indefinite integral for $f\big(y(x)\big)y'(x)$. Thus $\int f\big(y(x)\big)y'(x)\,dx = F\big(y(x)\big) + C$ or

$$\boxed{\int f\big(y(x)\big)y'(x)\,dx = \int f(y)\,dy\Big|_{y=y(x)}} \tag{S1}$$

This is the substitution rule for indefinite integrals. Note that, since $f\big(y(x)\big)y'(x)$, is a function of x, its indefinite integral must also be a function of x. On the right hand side, evaluating y at $y(x)$ ensures that we end up with a function of x.

Because $F\big(y(x)\big)$ is one indefinite integral of $f\big(y(x)\big)y'(x)$,

$$\int_a^b f\big(y(x)\big)y'(x)\,dx = F\big(y(x)\big)\Big|_{x=a}^{x=b} = F\big(y(b)\big) - F\big(y(a)\big)$$

The right hand side is $F(y) = \int f(y)\,dy$ evaluated at $y(b)$ minus the same function evaluated at $y(a)$. So

$$\boxed{\int_a^b f\big(y(x)\big)y'(x)\,dx = \int_{y(a)}^{y(b)} f(y)\,dy} \tag{S2}$$

This is the substitution rule for definite integrals. Notice that to get from the integral on the left hand side to the integral on the right hand side you

- substitute $y(x) \to y$ and $y'(x)dx \to dy$ (which looks like $\frac{dy}{dx} = y'(x)$ with the dx multiplied across)
- set the lower limit for the y integral to the value of y (namely $y(a)$) that corresponds to the lower limit of the x integral (namely $x = a$) and
- set the upper limit for the y integral to the value of y (namely $y(b)$) that corresponds to the upper limit of the x integral (namely $x = b$).

The substitution rule is used to simplify integrals, like $\int_0^\pi x^2 \sin\big(\frac{1}{3}x^3\big)\,dx$, in which the integrand

- has one factor $\big(\sin\big(\frac{1}{3}x^3\big)$ in this example$\big)$ which is some function $\big($sin in this example$\big)$ evaluated at some complicated argument $\big(\frac{1}{3}x^3$ in this example$\big)$ and
- has a second factor $\big(x^2$ in this example$\big)$ which is the derivative of the complicated argument, or at least a constant times the derivative of the complicated argument.

In this case one chooses $y(x)$ to be the complicated argument $\big($so $y(x) = \frac{1}{3}x^3$ in this example$\big)$.

Example 1 The integrand of

$$\int_0^1 e^x \sin\big(e^x\big)\,dx$$

is $e^x \sin\big(e^x\big)$. One factor of this integrand is $\sin\big(e^x\big)$, which is the function sin evaluated at e^x. The derivative of e^x is again e^x, which is the other factor in the integrand. Choose $y(x) = e^x$ and $f(y) = \sin y$. Then $f\big(y(x)\big) = \sin\big(e^x\big)$ and $y'(x) = e^x$ so

$$\int_0^1 e^x \sin\big(e^x\big)\,dx = \int_a^b f\big(y(x)\big)y'(x)\,dx$$

with $a = 0$ and $b = 1$. As $y(a) = y(0) = e^0 = 1$ and $y(b) = y(1) = e^1 = e$, the substitution rule gives

$$\int_0^1 e^x \sin\big(e^x\big)\,dx = \int_a^b f\big(y(x)\big)y'(x)\,dx = \int_{y(a)}^{y(b)} f(y)\,dy = \int_1^e \sin y\,dy = -\cos y\Big|_1^e = -\cos e + \cos 1$$

In conclusion

$$\boxed{\int_0^1 e^x \sin\big(e^x\big)\,dx = \cos 1 - \cos e}$$

Example 2 The integrand of

$$\int_0^1 x^2 \sin\big(x^3+1\big)\,dx$$

is $x^2\sin\big(x^3+1\big)$. One factor of this integrand is $\sin\big(x^3+1\big)$, which is the function sin evaluated at x^3+1. So set $y(x)=x^3+1$. The derivative $y'(x)=3x^2$ is not quite the other factor, x^2, in the integrand. But we can arrange for $y'(x)=3x^2$ to appear as a factor in the integrand just by multiplying and dividing by 3.

$$\int_0^1 x^2\sin\big(x^3+1\big)\,dx = \int_0^1 \tfrac{1}{3}\sin\big(x^3+1\big)\,3x^2dx$$

The integrand $\frac{1}{3}\sin\big(x^3+1\big)\,3x^2$ now is of the form $f\big(y(x)\big)y'(x)$ with $y(x)=x^3+1$ and $f(y)=\frac{1}{3}\sin y$. The limits of integration are $x=0$ and $x=1$. So, choosing $y(x)=x^3+1$, $f(y)=\frac{1}{3}\sin y$, $a=0$ and $b=1$ we have

$$\int_0^1 \tfrac{1}{3}\sin\big(x^3+1\big)\,3x^2dx = \int_a^b f\big(y(x)\big)y'(x)\,dx = \int_{y(a)}^{y(b)} f(y)\,dy = \int_1^2 \tfrac{1}{3}\sin y\,dy = -\tfrac{1}{3}\cos y\Big|_1^2 = \tfrac{-\cos 2}{3} - \tfrac{-\cos 1}{3}$$

In conclusion

$$\boxed{\int_0^1 \sin\big(x^3+1\big)\,x^2dx = \tfrac{\cos 1-\cos 2}{3}}$$

Once one has chosen $y(x)$, one can make the substitution without ever explicitly deciding what $f(y)$ is. One just has to note that the integrand on the right hand side of the substitution rule

$$\int_a^b f\big(y(x)\big)y'(x)\,dx = \int_{y(a)}^{y(b)} f(y)\,dy$$

is constructed from the integrand on the left hand side by

- substituting y for $y(x)$ and
- substituting dy for $y'(x)\,dx$

The substitution $dy=y'(x)\,dx$ is easily remembered by pretending that $\frac{dy}{dx}$ is an ordinary fraction. Then cross–multiplying $\frac{dy}{dx}=y'(x)$ gives $dy=y'(x)\,dx$.

Example 2 (revisited) Consider

$$\int_0^1 x^2\sin\big(x^3+1\big)\,dx$$

once again. We have observed that one factor of the integrand is $\sin\big(x^3+1\big)$, which is sin evaluated at x^3+1, and the other factor, x^2 is, aside from a factor of 3, the derivative of x^3+1. So we decide to try $y(x)=x^3+1$. Substitute y for x^3+1 and dy for $3x^2\,dx$. That is $x^3+1=y$ and $dy=3x^2\,dx$ or $x^2\,dx=\frac{dy}{3}$. When $x=0$, $y=0^3+1=1$. When $x=1$, $y=1^3+1=2$.

$$\int_0^1 \sin\big(x^3+1\big)\,x^2dx = \int_1^2 \sin y\,\tfrac{dy}{3}$$

We ended up with exactly this integral in example 2.

Example 3 $\int_0^{\pi/2}\cos(3x)\,dx$. Substitute for the argument of $\cos(3x)$. That, is $y(x)=3x$. We are to substitute $y=3x$ and $dy=3\,dx$ or $dx=\frac{dy}{3}$. When $x=0$, $y=3\times 0=0$. When $x=\frac{\pi}{2}$, $y=\frac{3}{2}\pi$.

$$\int_0^{\pi/2}\cos(3x)\,dx = \int_0^{3\pi/2}\cos(y)\,\tfrac{dy}{3} = \tfrac{\sin y}{3}\Big|_0^{3\pi/2} = \tfrac{-1}{3} - \tfrac{0}{3} = -\tfrac{1}{3}$$

Example 4 $\int_0^1 \frac{1}{(2x+1)^3}\,dx$. Substitute for the argument, $2x+1$, of $[2x+1]^{-3}$. That is, $y=2x+1$ and $dy=2\,dx$ or $dx=\frac{dy}{2}$. When $x=0$, $y=2\times 0+1=1$. When $x=1$, $y=2\times 1+1=3$.

$$\int_0^1 \frac{1}{(2x+1)^3}\,dx=\int_1^3 \frac{1}{y^3}\,\frac{dy}{2}=\frac{1}{2}\int_1^3 y^{-3}\,dy=\frac{1}{2}\frac{y^{-2}}{-2}\Big|_1^3=\frac{3^{-2}}{-4}-\frac{1^{-2}}{-4}=\frac{1}{4}\big[1-\frac{1}{9}\big]=\frac{2}{9}$$

Example 5 $\int_0^1 \frac{x}{1+x^2}\,dx$. Think of the integrand as the product $\frac{1}{1+x^2}x$. The first factor is the function "one over" evaluated at the argument $1+x^2$. The derivative of the argument $1+x^2$ is $2x$, which is, except for the 2, the second factor of the integrand. Substitute $y=1+x^2$, $dy=2x\,dx$ or $xdx=\frac{dy}{2}$. When $x=0$, $y=1+0^2=1$. When $x=1$, $y=1+1^2=2$.

$$\int_0^1 \frac{x}{1+x^2}\,dx=\int_1^2 \frac{1}{y}\,\frac{dy}{2}=\frac{1}{2}\ln|y|\Big|_1^2=\frac{\ln 2}{2}-\frac{0}{2}=\frac{1}{2}\ln 2$$

Example 6 $\int x^3\cos\big(x^4+2\big)\,dx$. The integrand is the product of cos evaluated at the argument x^4+2 times x^3, which aside from a factor of 4, is the derivative of the argument x^4+2. Substitute $y=x^4+2$, $dy=4x^3\,dx$ or $x^3dx=\frac{dy}{4}$.

$$\int x^3\cos\big(x^4+2\big)\,dx=\int \cos(y)\,\frac{dy}{4}=\frac{1}{4}\sin y+C$$

Because we are dealing with indefinite integrals we need not worry about limits of integration. On the other hand, $x^3\cos\big(x^4+2\big)$ is a function of x. So its indefinite integral $\big($which is defined to be a function whose derivative is $x^3\cos\big(x^4+2\big)$ $\big)$ must also be a function of x. The answer is $\frac{1}{4}\sin y(x)+C=\frac{1}{4}\sin\big(x^4+1\big)+C$. This is what (S1) says.

Example 7 $\int\sqrt{1+x^2}\,x^3\,dx$. Substitute for the argument of the square root. That is, substitute $y=1+x^2$, $dy=2x\,dx$ or $dx=\frac{dy}{2x}$. You might think that this does not eliminate all of the x's from $\sqrt{1+x^2}\,x^3\,dx=\sqrt{y}x^3\,\frac{dy}{2x}=\sqrt{y}x^2\,\frac{dy}{2}$. It does, provided you remember to substitute $x^2=y-1$ for the remaining factor of x^2.

$$\int\sqrt{1+x^2}x^3\,dx=\int\sqrt{y}(y-1)\,\frac{dy}{2}=\frac{1}{2}\int\big(y^{3/2}-y^{1/2}\big)\,dy=\frac{1}{2}\Big[\frac{y^{5/2}}{5/2}-\frac{y^{3/2}}{3/2}\Big]+C=\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$$

Don't forget to express the final answer in terms of x using $y=1+x^2$. Also, don't forget that you can always check that

$$\int\sqrt{1+x^2}x^3\,dx=\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$$

is correct. Just differentiate the right hand side

$$\begin{aligned}\frac{d}{dx}\Big[\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C\Big]&=\frac{1}{5}\frac{5}{2}(1+x^2)^{3/2}(2x)-\frac{1}{3}\frac{3}{2}(1+x^2)^{1/2}(2x)\\&=x(1+x^2)^{3/2}-x(1+x^2)^{1/2}=x\sqrt{1+x^2}\big[(1+x^2)-1\big]\\&=x\sqrt{1+x^2}\;x^2=x^3\sqrt{1+x^2}\end{aligned}$$

and verify that the answer is the same as the original integrand.

Example 8 $\int\tan x\,dx$. The secret here is to write the integrand $\tan x=\frac{1}{\cos x}\sin x$. Think of the first factor as the function "one over" evaluated at the argument $\cos x$. The derivative of the argument $\cos x$ is, except for a -1, the same as the second factor $\sin x$. Substitute $y=\cos x$, $dy=-\sin x\,dx$ or $\sin x\,dx=\frac{dy}{-1}$.

$$\int\tan x\,dx=\int\frac{1}{\cos x}\sin x\,dx=\int\frac{1}{y}\,\frac{dy}{-1}=-\ln|y|+C=-\ln|\cos x|+C=\ln|\cos x|^{-1}+C=\ln|\sec x|+C$$