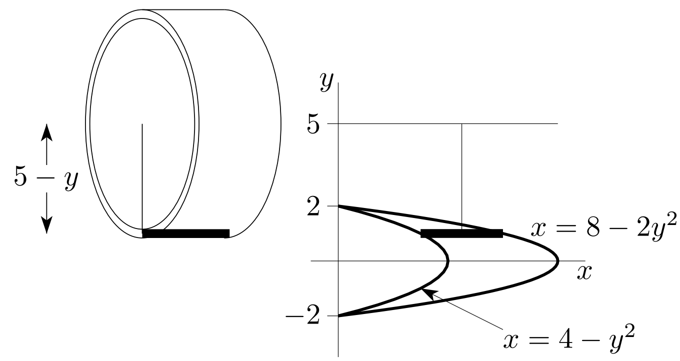
Cylindrical Shells Example

Find the volume of the solid obtained by rotating the region bounded by $x = 4 - y^2$ and $x = 8 - 2y^2$ about $y = 5$.

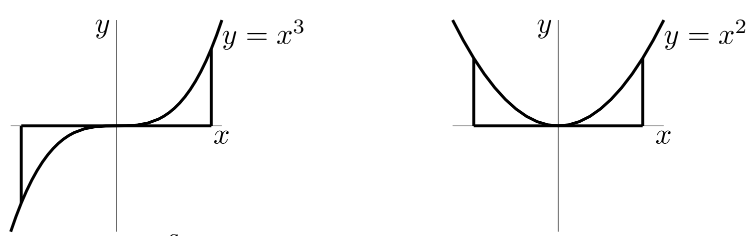
Solution. The region bounded by $x = 4 - y^2$ and $x = 8 - 2y^2$ is sketched below. Note that the two parabolas meet when $4 - y^2 = 8 - 2y^2$ or $y^2 = 4$ or $y = \pm 2$. The corresponding $x = 4 - (\pm 2)^2 = 0$. So, the two parabolas meet at $(0, \pm 2)$. Consider the thin slice in the figure on the right. It runs horizontally from $(4 - y^2, y)$ to $(8 - 2y^2, y)$ and has width dy. When this slice is rotated about $y = 5$, it sweeps out a cylindrical shell, as illustrated in the figure on the left. A radius for the shell is shown in the figure on the right. It is the vertical line half way along the thin slice. The y–coordinate of



the top end of the radius is 5 and the y–coordinate of the bottom end is y. So the radius has length $5 - y$. The height of the shell is the difference between the x–coordinates at the right and left hand ends of the thin slice. So the height of the shell is $(8 - 2y^2) - (4 - y^2) = 4 - y^2$. The thickness of the shell is dy and its volume is $2\pi(5 - y)(4 - y^2)dy$. The total volume of the solid is

$$\int_{-2}^{2} 2\pi(5-y)(4-y^2)dy = 2\pi \int_{-2}^{2} (20 - 4y - 5y^2 + y^3)\, dy$$

For any **odd power** y^n of y, and any a, the integral $\int_{-a}^{a} y^n\, dy = 0$. This is because the area with $-a \le y \le 0$ has the same magnitude but opposite sign as the area with $0 \le y \le a$. See the figure on the left below. Thus the integrals $\int_{-2}^{2} y\, dy = \int_{-2}^{2} y^3\, dy = 0$. For any **even power** y^n of



y, and any a, the integral $\int_{-a}^{a} y^n\, dy = 2\int_{0}^{a} y^n\, dy$. This is because the area with $-a \le y \le 0$ has the same magnitude and same sign as the area with $0 \le y \le a$. See the figure on the right above.

The volume of the solid is

$$2\pi \int_{-2}^{2} (20 - 4y - 5y^2 + y^3)\, dy = 2\pi \int_{-2}^{2} (20 - 5y^2)\, dy = 4\pi \int_{0}^{2} (20 - 5y^2)\, dy$$

$$= 20\pi \int_{0}^{2} (4 - y^2)\, dy = 20\pi \left[4y - \tfrac{1}{3}y^3\right]_0^2 = 20\pi \left[8 - \tfrac{8}{3}\right] = \boxed{\tfrac{320}{3}\pi}$$