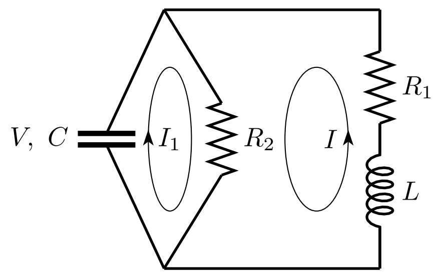
IV. Eigenvalues and Eigenvectors

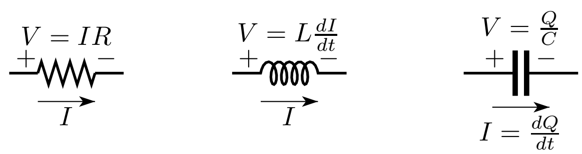
§IV.1 An Electric Circuit – Equations

Consider the electric circuit



The following are the experimental facts of life that determine the voltages across and currents through resistors, capacitors and inductances:

- The voltage across a resistor of resistance R is IR, where I is the current flowing through the resistor.
- The voltage across a capacitor of capacitance C is Q/C, where Q is the charge on the capacitor.
- The current through a capacitor is $\frac{dQ}{dt}$, where Q is the charge on the capacitor.
- The voltage across an inductor of inductance L is $L\frac{dI}{dt}$, where I is the current flowing through the inductor.



The currents and voltages of a circuit built, as in the above example, out of a number of circuit elements are determined by two other experimental facts of life, called Kirchhoff's laws. They are

- The voltage between any two points of the circuit is independent of the path used to travel between the two points.
- The net current entering any given node of the circuit is zero. As we have already observed, in Example II.9, if one uses current loops, as in the figure above, Kirchhoff's current law is automatically satisfied.

Let, for the above circuit,

$$
\begin{aligned}
Q &= \text{the charge on } C \\
V &= \tfrac{Q}{C} = \text{the voltage across } C \\
I &= \text{the loop current through } L \text{ as in the figure above} \\
I_1 &= \tfrac{dQ}{dt} = \text{the loop current through } C \text{ as in the figure above}
\end{aligned}
$$

By Kirchhoff's voltage law, applied to the two loops in the figure above,

$$
\begin{aligned}
\tfrac{Q}{C} + R_2(I_1 + I) &= 0 \\
L\frac{dI}{dt} + IR_1 + R_2(I_1 + I) &= 0
\end{aligned}
$$

Generally the voltage across a capacitor is of greater interest than the charge on the capacitor. So let's substitute $Q = CV$ and $I_1 = \frac{dQ}{dt} = C\frac{dV}{dt}$.

$$
\begin{aligned}
V + R_2\big(C\tfrac{dV}{dt} + I\big) &= 0 \\
L\frac{dI}{dt} + IR_1 + R_2\big(C\tfrac{dV}{dt} + I\big) &= 0
\end{aligned}
$$

We can also substitute $R_2\big(C\frac{dV}{dt}+I\big)=-V$, from the first equation, into the second equation. So

$$\begin{aligned} V + R_2\big(C\tfrac{dV}{dt}+I\big) &= 0 \\ L\frac{dI}{dt} + IR_1 - V &= 0 \end{aligned}$$

or

$$\begin{aligned} \frac{dI}{dt} &= -\frac{R_1}{L}I + \frac{1}{L}V \\ \frac{dV}{dt} &= -\frac{1}{C}I - \frac{1}{R_2C}V \end{aligned}$$

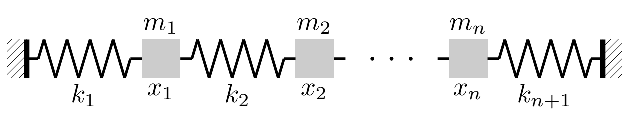
This is an example of a system of linear first order ordinary differential equations. Furthermore the equations are homogeneous and have constant coefficients. The significance of each of these adjectives is

- ordinary: The unknowns I and V are functions of a single variable t. Consequently, all derivatives of these unknowns are ordinary derivatives $\frac{d}{dt}$ rather than partial derivatives $\frac{\partial}{\partial t}$.
- differential: The equations involve derivatives of the unknown functions.
- first order: The order of the highest derivative that appears is one. That is, no $\frac{d^n}{dt^n}$ with $n>1$ appears.
- linear: Each term in the equations is either independent of the unknown functions or is proportional to the first power of an unknown function (possibly differentiated).
- constant coefficient: Each term in the equations is either independent of the unknowns or is a constant times the first power of an unknown (possibly differentiated).
- homogeneous: There are no terms in the equations that are independent of the unknowns.

In this chapter, we shall learn how to solve such systems of linear first order ordinary differential equations.

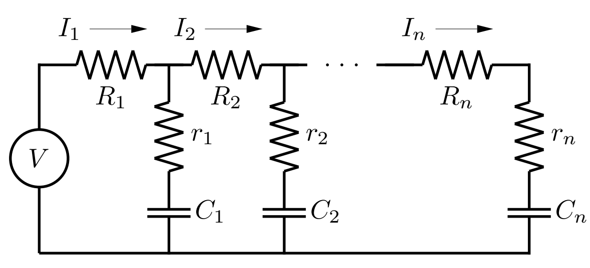
Exercises for §IV.1.

1) Consider a system of n masses coupled by springs as in the figure



The masses are constrained to move horizontally. The distance from mass number j to the left hand wall is x_j and its mass is m_j. The $j^{\rm th}$ spring has natural length ℓ_j and spring constant k_j. This means that the force exerted by spring number j is k_j times the extension of spring number j, where the extension of a spring is its length minus its natural length. The distance between the two walls is L. Problem 1 of §II.5 asked for the system of equations that determined the equilibrium values of $x_1,\cdots,x_j$. Now let the masses to move. Write down Newton's law of motion for $x_1(t),\cdots,x_j(t)$.

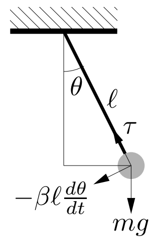
2) Consider the electrical network in the figure



Assume that the voltage $V(t)$ is given, that the resistances $R_1,\cdots,R_n$ and $r_1,\cdots,r_n$ are given and that the capacitances $C_1,\cdots,C_n$ are given. Find the system of equations that determine the currents $I_1,\cdots,I_n$.

§IV.2 The Pendulum – Equations

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ. Denote by θ the angle between the rod and vertical. The forces acting on the



mass are gravity, which has magnitude mg and direction $(0,-1)$, tension in the rod, whose magnitude $\tau(t)$ automatically adjusts itself so that the distance between the mass and the hinge is fixed at ℓ and whose direction is always parallel to the rod and possibly some frictional forces, like friction in the hinge and air resistance. Assume that the total frictional force has magnitude proportional to the speed of the mass and has direction opposite to the direction of motion of the mass.

We have already seen in Chapter I that this pendulum obeys

$$m\ell\tfrac{d^2\theta}{dt^2} = -mg\sin\theta - \beta\ell\tfrac{d\theta}{dt}$$

and that when θ is small, we can approximate $\sin\theta \approx \theta$ and get the equation

$$\tfrac{d^2\theta}{dt^2} + \tfrac{\beta}{m}\tfrac{d\theta}{dt} + \tfrac{g}{\ell}\theta = 0$$

We can reformulate this second order linear ordinary differential equation by a system of first order equations simply by introducing the second unknown

$$s = \tfrac{d\theta}{dt}$$

Then the second order derivative $\frac{d^2\theta}{dt^2}$ can be eliminated by replacing it with $\frac{ds}{dt}$.

$$\begin{aligned} \tfrac{d\theta}{dt} &= s \\ \tfrac{ds}{dt} &= -\tfrac{g}{\ell}\theta - \tfrac{\beta}{m}s \end{aligned}$$

§IV.3 Systems of First Order Constant Coefficient Homogeneous Ordinary Differential Equations

Definition IV.1 A system of first order constant coefficient homogeneous ordinary differential equations (ODE's) is a family of n ODE's in n unknown functions $x_1(t),\ \cdots,\ x_n(t)$ that can be written in the form

$$\frac{d\vec{x}}{dt} = A\vec{x}(t)$$

where $\vec{x}$ is the column vector whose i^{th} row is $x_i(t)$ and A is an $n \times n$ matrix with entries that are constants independent of t.

Systems of ODE's tend to arise by some combination of two basic mechanisms. First, the original problem may involve the rates of change of more than one quantity. For example, in linear circuit problems one studies the behaviour of complex electrical circuits built from "linear circuit elements" like resistors, capacitors and inductances. The unknowns are the currents I_ℓ in the various branches of the circuit and the charges $Q_j(t)$ on the various capacitors. The equations come from Kirchhoff's laws that state the total voltage around any closed loop in the circuit must be zero and that the total current entering any node of the circuit must be zero. We have seen an example in §IV.1.

The second mechanism is the conversion of one higher order equation

$$\tfrac{d^n x}{dt^n}(t) = F\big(x(t), x'(t), \cdots, \tfrac{d^{n-1}x}{dt^{n-1}}(t)\big)$$

into a system of first order equations by the simple expedient of viewing each of the first $n-1$ derivatives as a different unknown function.

$$x_j(t) = \tfrac{d^j x}{dt^j}(t), \qquad 0 \le j \le n-1$$

For each $0 \le j \le n-2$

$$\tfrac{d}{dt}x_j(t) = \tfrac{d}{dt}\tfrac{d^j}{dt^j}x(t) = \tfrac{d^{j+1}}{dt^{j+1}}x(t) = x_{j+1}(t)$$

and for $j = n-1$

$$\begin{aligned}\tfrac{d}{dt}x_{n-1}(t) &= \tfrac{d}{dt}\tfrac{d^{n-1}}{dt^{n-1}}x(t) = \tfrac{d^n x}{dt^n}(t) = F\big(x(t), x'(t), \cdots, \tfrac{d^{n-1}x}{dt^{n-1}}(t)\big)\\ &= F\big(x_0(t), x_1(t), \cdots, x_{n-1}(t)\big)\end{aligned}$$

So, the system

$$\begin{aligned}\tfrac{d}{dt}x_0(t) &= x_1(t)\\ \vdots \quad &\qquad \vdots\\ \tfrac{d}{dt}x_{n-2}(t) &= x_{n-1}(t)\\ \tfrac{d}{dt}x_{n-1}(t) &= F\big(x_0(t), x_1(t), \cdots, x_{n-1}(t)\big)\end{aligned}$$

is equivalent to the original higher order system. That is, for each solution of the higher order equation, there is a corresponding solution of the first order system and vice versa. We have seen an example of this mechanism in the pendulum problem of §IV.2.

We next attempt to solve

$$\frac{d\vec{x}}{dt} = A\vec{x}(t)$$

simply by guessing. Recall that $\vec{x}(t)$ is an unknown function of time. For each different value of t, $\vec{x}(t)$ is a different unknown variable. So we really have infinitely many equations in infinitely many unknowns. We shall make a guess such that $\vec{x}\,'(t)$ and $A\vec{x}(t)$ have the same time dependence. That is, such that $\vec{x}\,'(t)$ is proportional to $\vec{x}(t)$. Then all t's will cancel out of the equation. The one function whose derivative is proportional to itself is the exponential, so we guess $\vec{x}(t) = e^{\lambda t}\vec{v}$ where λ and $\vec{v}$ are constants to be chosen so as to give a solution. Our guess is a solution if and only if

$$\frac{d}{dt}\left(e^{\lambda t}\vec{v}\right) = A\left(e^{\lambda t}\vec{v}\right)$$

or equivalently

$$\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}$$

Because the derivative of an exponential is proportional to the same exponential, both terms in the equation are proportional to the same exponential and we can eliminate all t dependence from the equation just by dividing it by $e^{\lambda t}$.

$$\lambda\vec{v} = A\vec{v}$$

As a result, we have to solve for $n+1$ unknowns λ, $\vec{v}$ rather than for infinitely many unknowns in the form of n unknown functions, $\vec{x}(t)$. For any λ, $\vec{v} = \vec{0}$ always a solution. In other words $\vec{x}(t) = \vec{0}$ is always a solution of $\frac{d\vec{x}}{dt} = A\vec{x}(t)$. This solution is pretty useless and is called the trivial solution. We really want nontrivial solutions. We find them in the next section.

Exercises for §IV.3.

1) Convert each of the following higher order differential equations into a system of first order equations.
   a) $y'' + 6y' + 5y = 0$ b) $y^{(6)} - 16y = 0$
2) Convert each of the following systems of first order differential equations into a single higher order equation.
   a) $\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$ b) $\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x}$

§IV.4 Eigenvalues and Eigenvectors

Definition IV.2 Let A be a matrix. An eigenvector of A of eigenvalue λ is a **nonzero** vector $\vec{v}$ that obeys

$$A\vec{v} = \lambda\vec{v}$$

Note that any nonzero linear combination $s\vec{u} + t\vec{v}$ of eigenvectors of A with (**the same**) eigenvalue λ is again an eigenvector of A with eigenvalue λ because

$$\begin{aligned} A(s\vec{u} + t\vec{v}) &= sA\vec{u} + tA\vec{v} \\ &= s\lambda\vec{u} + t\lambda\vec{v} \\ &= \lambda(s\vec{u} + t\vec{v}) \end{aligned}$$

First, let's concentrate on the problem of determining the unknowns $\vec{v}$ once we know the value of λ. Once λ is known, we are left with a system of linear equations in the unknowns $\vec{v}$. We can write this system in standard form (i.e. a matrix times $\vec{v}$ equals a constant vector) by recalling, from §III.6 on inverse matrices, that $\lambda\vec{v} = \lambda I\vec{v}$ where I is the $n \times n$ matrix which has zero in all of its off-diagonal entries and 1 in all of its diagonal entries.

$$A\vec{v} = \lambda\vec{v} \iff A\vec{v} = \lambda I\vec{v} \iff A\vec{v} - \lambda I\vec{v} = \vec{0} \iff (A - \lambda I)\vec{v} = \vec{0}$$

This is a linear homogeneous system of equations with coefficient matrix $A - \lambda I$. We know that $\vec{v} = \vec{0}$ is always a solution. We also know that there is exactly one solution (in this case $\vec{0}$) if and only if the determinant $\det(A - \lambda I) \neq 0$. Consequently

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff (A - \lambda I)\vec{v} = \vec{0} \text{ has a nonzero solution} \\ &\iff \det(A - \lambda I) = 0 \\ &\iff \lambda \text{ is a root of } C_A(\lambda) = \det(A - \lambda I) \end{aligned}$$

Once we have determined the eigenvalues, that is, the roots of $C_A(\lambda)$, which is called the characteristic polynomial of A, we can find all nontrivial (i.e. not $\vec{0}$) solutions of $(A - \lambda I)\vec{v} = \vec{0}$, that is all eigenvectors, by Gaussian elimination.

Example IV.3 Let

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Then λ is an eigenvalue of A if and only if

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det \begin{bmatrix} -2 - \lambda & 1 \\ 1 & -2 - \lambda \end{bmatrix} \\ &= (-2 - \lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) \end{aligned}$$

The eigenvalues are -1 and -3. The eigenvectors of eigenvalue -1 are all nonzero solutions of

$$\begin{bmatrix} -2-(-1) & 1 \\ 1 & -2-(-1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{namely} \qquad \vec{v} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ c \neq 0$$

Similarly, the eigenvectors of eigenvalue -3 are all nonzero solutions of

$$\begin{bmatrix} -2-(-3) & 1 \\ 1 & -2-(-3) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{namely} \qquad \vec{v} = c \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ c \neq 0$$

To check that these are correct, we just have to verify that

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example IV.4 Let

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

Then λ is an eigenvalue of A if and only if

$$\begin{aligned}
0 = \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{bmatrix} \\
&= (3-\lambda)\det \begin{bmatrix} 4-\lambda & 1 \\ -4 & -1-\lambda \end{bmatrix} - 2\det \begin{bmatrix} 1 & 1 \\ -2 & -1-\lambda \end{bmatrix} + 2\det \begin{bmatrix} 1 & 4-\lambda \\ -2 & -4 \end{bmatrix} \\
&= (3-\lambda)[(4-\lambda)(-1-\lambda)+4] - 2[-1-\lambda+2] + 2[-4+2(4-\lambda)] \\
&= (3-\lambda)[\lambda^2 - 3\lambda - 4 + 4] - 2\lambda + 6 = (3-\lambda)[\lambda^2 - 3\lambda + 2] \qquad (\text{we wrote } -2\lambda + 6 = 2(3-\lambda)) \\
&= (3-\lambda)(\lambda-2)(\lambda-1)
\end{aligned}$$

The eigenvalues are 1, 2 and 3. Here we were able to simplify the problem of finding the roots of $\det(A-\lambda I)$ by recognizing that $(\lambda - 3)$ was a factor relatively early in the computation. Had we not kept $(3-\lambda)$ as a factor, we would have found that $\det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$. Some useful tricks for finding roots of poynomials like this are given in Appendix IV.A. In particular, those tricks are used in Example IV.A.4 to find the roots of $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$.

The eigenvectors of eigenvalue 1 are all nonzero solutions of

$$\begin{bmatrix} 3-1 & 2 & 2 \\ 1 & 4-1 & 1 \\ -2 & -4 & -1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reducing,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \begin{array}{c} (1)/2 \\ (2)-(1)/2 \\ (3)+(1) \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3)+(2) \end{array} \qquad \text{gives } \vec{v} = c \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},\ c \neq 0$$

Similarly, the eigenvectors of eigenvalue 2 are all nonzero solutions of

$$\begin{bmatrix} 3-2 & 2 & 2 \\ 1 & 4-2 & 1 \\ -2 & -4 & -1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reducing,

$$\left[\begin{array}{ccc|c}1&2&2&0\\0&0&1&0\\0&0&1&0\end{array}\right]\begin{array}{c}(1)\\(2)-(1)\\(3)+2(1)\end{array}\qquad\left[\begin{array}{ccc|c}1&2&2&0\\0&0&1&0\\0&0&0&0\end{array}\right]\begin{array}{c}(1)\\(2)\\(3)-(2)\end{array}\qquad\text{gives }\vec v=c\begin{bmatrix}2\\-1\\0\end{bmatrix},\ c\neq 0$$

Finally, the eigenvectors of eigenvalue 3 are all nonzero solutions of

$$\begin{bmatrix}3-3&2&2\\1&4-3&1\\-2&-4&-1-3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0&2&2\\1&1&1\\-2&-4&-4\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Row reducing,

$$\left[\begin{array}{ccc|c}1&1&1&0\\0&1&1&0\\0&-2&-2&0\end{array}\right]\begin{array}{c}(2)\\(1)/2\\(3)+2(2)\end{array}\qquad\left[\begin{array}{ccc|c}1&1&1&0\\0&1&1&0\\0&0&0&0\end{array}\right]\begin{array}{c}(1)\\(2)\\(3)+2(2)\end{array}\qquad\text{gives }\vec v=c\begin{bmatrix}0\\1\\-1\end{bmatrix},\ c\neq 0$$

To check that these are correct, observe that

$$\begin{bmatrix}3&2&2\\1&4&1\\-2&-4&-1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=1\begin{bmatrix}1\\0\\-1\end{bmatrix},\quad\begin{bmatrix}3&2&2\\1&4&1\\-2&-4&-1\end{bmatrix}\begin{bmatrix}2\\-1\\0\end{bmatrix}=2\begin{bmatrix}2\\-1\\0\end{bmatrix},$$
$$\begin{bmatrix}3&2&2\\1&4&1\\-2&-4&-1\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix}=3\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Example IV.5 Let

$$A=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}$$

Then λ is an eigenvalue of A if and only if

$$\begin{aligned}0=\det(A-\lambda I)&=\det\begin{bmatrix}3-\lambda&2&4\\2&-\lambda&2\\4&2&3-\lambda\end{bmatrix}\\
&=(3-\lambda)\det\begin{bmatrix}-\lambda&2\\2&3-\lambda\end{bmatrix}-2\det\begin{bmatrix}2&2\\4&3-\lambda\end{bmatrix}+4\det\begin{bmatrix}2&-\lambda\\4&2\end{bmatrix}\\
&=(3-\lambda)[-\lambda(3-\lambda)-4]-2[6-2\lambda-8]+4[4+4\lambda]\\
&=(3-\lambda)[\lambda^2-3\lambda-4]+20\lambda+20=(3-\lambda)(\lambda-4)(\lambda+1)+20(\lambda+1)\\
&=(\lambda+1)[(3-\lambda)(\lambda-4)+20]=(\lambda+1)[-\lambda^2+7\lambda+8]=(\lambda+1)(-\lambda+8)(\lambda+1)\end{aligned}$$

The eigenvalues are -1 (which is said to have multiplicity two, because it is a double root of the characteristic polynomial $(\lambda+1)(-\lambda+8)(\lambda+1)$) and 8. The eigenvectors of eigenvalue 8 are all nonzero solutions of

$$\begin{bmatrix}3-8&2&4\\2&-8&2\\4&2&3-8\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}-5&2&4\\2&-8&2\\4&2&-5\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Row reducing,

$$\left[\begin{array}{ccc|c}1&-4&1&0\\0&-18&9&0\\0&18&-9&0\end{array}\right]\begin{array}{c}(2)/2\\(1)+5(2)/2\\(3)-2(2)\end{array}\qquad\left[\begin{array}{ccc|c}1&-4&1&0\\0&-2&1&0\\0&0&0&0\end{array}\right]\begin{array}{c}(1)\\(2)/9\\(3)+(2)\end{array}\qquad\text{gives }\vec v=c\begin{bmatrix}2\\1\\2\end{bmatrix},\ c\neq 0$$

The eigenvectors of eigenvalue -1 are all nonzero solutions of

$$\begin{bmatrix} 3+1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 3+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reducing,

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} (1)/2 \\ (2)-(1)/2 \\ (3)-(1) \end{array} \qquad \text{gives } \vec{v} = c \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix},\ c, d \text{ not both zero}$$

Checking,

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 8 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Observe that in this case our eigenvalue of multiplicity two had two "really different" eigenvectors in the sense that

$$\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \neq c \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

for all values of c. So, in writing down the general solution, we cannot absorb $[0, -2, 1]$ in $c[1, -2, 0]$ or vice versa. This is typical, but not universal (as we shall see in §IV.8), behaviour when there are repeated eigenvalues.

Exercises for §IV.4

1) Find all eigenvalues and eigenvectors of each of the following matrices.

$$a)\ \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \qquad b)\ \begin{bmatrix} -2 & -8 \\ 4 & 10 \end{bmatrix} \qquad c)\ \begin{bmatrix} 29 & -10 \\ 105 & -36 \end{bmatrix} \qquad d)\ \begin{bmatrix} -9 & -14 \\ 7 & 12 \end{bmatrix}$$

2) Find all eigenvalues and eigenvectors of each of the following matrices.

$$a)\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & 2 \end{bmatrix} \qquad b)\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \qquad c)\ \begin{bmatrix} 7 & -9 & -15 \\ 0 & 4 & 0 \\ 3 & -9 & -11 \end{bmatrix} \qquad d)\ \begin{bmatrix} 31 & -100 & 70 \\ 18 & -59 & 42 \\ 12 & -40 & 29 \end{bmatrix}$$

3) Find all eigenvalues and eigenvectors of each of the following matrices, without determining explicitly what the matrix is.
   a) A is a 2×2 matrix that projects onto the line $x + y = 0$.
   b) B is a 2×2 matrix that reflects in the line $x + y = 0$.
   c) C is a 3×3 matrix that reflects in the plane $x + 2y + 3z = 0$.

§IV.5 An Electric Circuit – Solution

Example IV.6 Consider the electric circuit of §IV.1 with $C = \frac{2}{3}$, $R_1 = 1$, $R_2 = \frac{3}{5}$ and $L = 2$. These numbers are chosen to make the numbers in the solution work out nicely. The current I and voltage V then obey

$$\begin{aligned} \tfrac{dI}{dt} &= -\tfrac{1}{2}I + \tfrac{1}{2}V \\ \tfrac{dV}{dt} &= -\tfrac{3}{2}I - \tfrac{5}{2}V \end{aligned} \qquad \text{or} \qquad \frac{d}{dt}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix} \qquad \text{or} \qquad \frac{d\vec{x}}{dt} = A\vec{x}$$

with

$$\vec{x} = \begin{bmatrix} I \\ V \end{bmatrix} \qquad\qquad A = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

Try $\vec{x}(t) = e^{\lambda t}\vec{v}$, with the constants λ and $\vec{v}$ to be determined. This guess is a solution if and only if

$$\lambda e^{\lambda t}\vec{v} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix} e^{\lambda t}\vec{v}$$

Dividing both side of the equation by $e^{\lambda t}$ gives

$$\lambda\vec{v} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}\vec{v} \quad \text{or} \quad \left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)\vec{v} = \vec{0} \quad \text{or} \quad \begin{bmatrix} -\frac{1}{2} - \lambda & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} - \lambda \end{bmatrix}\vec{v} = \vec{0}$$

This system of equations always has the trivial solution $\vec{v} = \vec{0}$. It has a solution with $\vec{v} \neq \vec{0}$ if and only if

$$\det \begin{bmatrix} -\frac{1}{2} - \lambda & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} - \lambda \end{bmatrix} = 0$$

Evaluating the determinant

$$\left(-\tfrac{1}{2} - \lambda\right)\left(-\tfrac{5}{2} - \lambda\right) + \tfrac{1}{2} \times \tfrac{3}{2} = 0$$

and simplifying

$$\lambda^2 + 3\lambda + \tfrac{5}{4} + \tfrac{3}{4} = (\lambda + 1)(\lambda + 2) = 0$$

we conclude that the eigenvalues of A are -1 and -2.

The eigenvectors of A of eigenvalue -1 consist of all nonzero solutions of

$$\begin{bmatrix} -\frac{1}{2} - (-1) & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} - (-1) \end{bmatrix}\vec{v} = \vec{0}$$

Simplifying, applying Gaussian elimination and backsolving

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}\vec{v} = \vec{0} \qquad \begin{array}{r} (1) \\ (2) + 3(1) \end{array} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \qquad \begin{array}{l} v_2 = c_1, \text{ arbitrary} \\ v_1 = -v_2 = -c_1 \end{array} \qquad \vec{v} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Similarly, the eigenvectors of A of eigenvalue -2 consist of all nonzero solutions of

$$\begin{bmatrix} -\frac{1}{2} - (-2) & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} - (-2) \end{bmatrix}\vec{v} = \vec{0}$$

Again simplifying, applying Gaussian elimination and backsolving

$$\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}\vec{v} = \vec{0} \qquad \begin{array}{r} (1) \\ (2) + (1) \end{array} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \qquad \begin{array}{l} v_2 = c_2, \text{ arbitrary} \\ v_1 = -\frac{1}{3}v_2 = -\frac{1}{3}c_2 \end{array} \qquad \vec{v} = c_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Note that, for both $\lambda = -1$ and $\lambda = -2$, the matrix resulting from applying Gaussian elimination to $A - \lambda I$ (i.e. the second matrix in each of the two above computations) has a row of zeros. This ensures that there is a nonzero solution $\vec{v}$. We know that there must be nonzero solutions, because $\det(A - \lambda I) = 0$ for both $\lambda = -1$ and $\lambda = -2$. If, in computing eigenvectors, you do not find a row of zeros after performing Gaussian elimination, then the only solution is $\vec{v} = \vec{0}$ (which is not a legal eigenvector) and you **must** have made a mechanical error somewhere.

We started this example looking for solutions of $\frac{d\vec{x}}{dt} = A\vec{x}(t)$ of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$. We have found that

$$\lambda = -1 \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad\qquad \lambda = -2 \quad \vec{v} = \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

both give solutions. Note that, if $\vec{u}(t)$ and $\vec{v}(t)$ both solve $\frac{d\vec{x}}{dt} = A\vec{x}(t)$, then so does the linear combination $c_1\vec{u}(t) + c_2\vec{v}(t)$ for any values of the constants c_1 and c_2 because

$$\begin{array}{rcl} \frac{d}{dt}\big(c_1\vec{u}(t) + c_2\vec{v}(t)\big) &=& c_1\ \frac{d\vec{u}(t)}{dt} + c_2\ \frac{d\vec{v}(t)}{dt} \\ && \qquad \| \qquad\quad \| \\ A\big(c_1\vec{u}(t) + c_2\vec{v}(t)\big) &=& c_1\ A\vec{u}(t) + c_2\ Av(t) \end{array} \tag{IV.1}$$

So we conclude that

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}-1/3\\1\end{bmatrix} \quad\Longrightarrow\quad \frac{d\vec{x}}{dt} = \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & -\frac{5}{2}\end{bmatrix}\vec{x}(t)$$

At this stage we have a two parameter family of solutions to the differential equation. (We shall see in §IV.9 that there aren't any other solutions.) The values of the parameters c_1 and c_2 cannot be determined by the differential equation itself. Usually they are determined by initial conditions. Suppose, for example, that we are told that

$$\vec{x}(0) = \begin{bmatrix}I(0)\\V(0)\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

In order for our solution to satisfy this iniitial condition, the parameters c_1, c_2 must obey

$$\begin{bmatrix}0\\2\end{bmatrix} = \vec{x}(0) = c_1 e^{-0}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{-2\times 0}\begin{bmatrix}-1/3\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\1\end{bmatrix} + c_2\begin{bmatrix}-1/3\\1\end{bmatrix} = \begin{bmatrix}-1 & -1/3\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

This is a linear system of equations that can be solved by Gaussian elimination, or by simply observing that the first equation forces $c_2 = -3c_1$ and the second equation forces $c_1 + c_2 = 2$ so that $c_1 = -1$ and $c_2 = 3$. So the solution of the initial value problem

$$\frac{d\vec{x}}{dt} = \begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{3}{2} & -\frac{5}{2}\end{bmatrix}\vec{x}(t), \quad \vec{x}(0) = \begin{bmatrix}0\\2\end{bmatrix}$$

is

$$\vec{x}(t) = -e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + 3e^{-2t}\begin{bmatrix}-1/3\\1\end{bmatrix} = -e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + e^{-2t}\begin{bmatrix}-1\\3\end{bmatrix}$$

With this choice of initial conditions, we are starting with a charge on the capacitor (since $V(0) = 2$) and no current flowing (since $I(0) = 0$). Our solution shows that as time progresses the capacitor discharges exponentially quickly through the circuit.

Example IV.7 Once again consider the electric circuit of §IV.1 but this time with $C = R_1 = R_2 = L = 1$. The current I and voltage V then obey

$$\begin{array}{l}\frac{dI}{dt} = -I + V\\ \frac{dV}{dt} = -I - V\end{array} \qquad\text{or}\qquad \frac{d}{dt}\vec{x}(t) = A\ \vec{x}(t) \qquad\text{with}\qquad A = \begin{bmatrix}-1 & 1\\-1 & -1\end{bmatrix},\ \vec{x} = \begin{bmatrix}I\\V\end{bmatrix}$$

We first find the eigenvalues, of course.

$$\det(A-\lambda I) = \det\begin{bmatrix}-1-\lambda & 1\\ -1 & -1-\lambda\end{bmatrix} = (\lambda+1)^2+1 = 0 \iff (\lambda+1)^2 = -1 \iff \lambda+1 = \pm i \iff \lambda = -1\pm i$$

so the eigenvalues of A are $-1+i$ and $-1-i$, with i being the "imaginary" number $\sqrt{-1}$. Despite their (inappropriate) name, imaginary numbers arise over and over in the real world. If you are not comfortable with them, you should review their definition and properties now. These are given in Appendix B.

The eigenvectors of eigenvalue $\lambda = -1 + i$ are the nonzero solutions of

$$(A - \lambda I)\big|_{\lambda=-1-i}\vec v = \vec 0 \iff \det \begin{bmatrix} -1-(-1+i) & 1 \\ -1 & -1-(-1+i) \end{bmatrix} \vec v = \vec 0$$

$$\iff \det \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \vec v = \vec 0 \iff \vec v = c \begin{bmatrix} 1 \\ i \end{bmatrix},\ c \neq 0$$

We could now repeat this (easy) computation with $\lambda = -1 - i$. But there's an even easier way. If A is any square matrix with real entries (as is the case in this example) and $\vec v$ is an eigenvector of A with complex eigenvalue λ, then, by definition

$$A\vec v = \lambda \vec v$$

Take the complex conjugate of both sides. Since A has real entries, the complex conjugate of A is again A. So

$$A\,\overline{\vec v} = \bar\lambda\,\overline{\vec v}$$

As $\vec v$ is a nonzero vector, this says, by definition, that the complex conjugate of $\vec v$ is an eigenvector of A of eigenvalue $\bar\lambda$. So we may conclude, without any computation, that, in this example, the eigenvectors of eigenvalue $-1 - i = \overline{-1+i}$ are

$$\vec v = c \begin{bmatrix} 1 \\ -i \end{bmatrix},\ c \neq 0$$

At this point, we have found that

$$e^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \text{and} \qquad e^{(-1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

both solve $\frac{d}{dt}\vec x = A\vec x$. By the linearity argument of (IV.1),

$$\vec x(t) = c_1 e^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(-1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{obeys} \qquad \frac{d\vec x}{dt} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\vec x(t)$$

for all c_1 and c_2. This is in fact the general solution. (An argument justifying this is given in §CH.9.)

At first site, this general solution looks bizarre. It is complex, while $I(t)$ and $V(t)$ are both certainly real quantities. Here is why this is not a contradiction. When one chooses a real valued initial condition, the constants c_1 and c_2 necessarily take values (*note that c_1 and c_2 are allowed to be complex*) such that $\vec x(t)$ is also real valued. Here is an example. Suppose that, as in Example IV.6,

$$\vec x(0) = \begin{bmatrix} I(0) \\ V(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

In order for our solution to satisfy this iniitial condition, the parameters c_1, c_2 must obey

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \vec x(0) = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The first equation forces $c_2 = -c_1$ and the second equation forces $ic_1 - ic_2 = 2$ so that $2ic_1 = 2$ and $c_1 = -i$, $c_2 = i$. So the solution of the initial value problem

$$\frac{d\vec x}{dt} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\vec x(t), \quad \vec x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

is

$$\vec x(t) = -ie^{(-1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + ie^{(-1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

While this still looks complex, it isn't, because the two terms are complex conjugates of each other. It is not hard to simplify the answer and eliminate all $\sqrt{-1}$'s:

$$\vec{x}(t) = -ie^{(-1+i)t}\begin{bmatrix}1\\ i\end{bmatrix} + ie^{(-1-i)t}\begin{bmatrix}1\\ -i\end{bmatrix} = e^{-t}\begin{bmatrix}-i(e^{it}-e^{-it})\\ e^{it}+e^{-it}\end{bmatrix}$$

Now we just either have to remember that $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ or remember that $\sin t = \frac{1}{2i}\big(e^{it} - e^{-it}\big)$ and $\cos t = \frac{1}{2}\big(e^{it} + e^{-it}\big)$. Using either gives

$$\vec{x}(t) = 2e^{-t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

With this choice of initial conditions, we are again starting with a charge on the capacitor (since $V(0) = 2$) and no current flowing (since $I(0) = 0$). This time while the capacitor discharges exponentially quickly through the circuit, the voltage oscillates.

Example IV.8 Let's return to the general solution

$$\vec{x}(t) = c_1e^{(-1+i)t}\begin{bmatrix}1\\ i\end{bmatrix} + c_2e^{(-1-i)t}\begin{bmatrix}1\\ -i\end{bmatrix}$$

of Example IV.7. In this example we shall see that it is possible to rewrite it so that no $\sqrt{-1}$ appears. Start by writing

$$\vec{v}_1(t) = e^{(-1+i)t}\begin{bmatrix}1\\ i\end{bmatrix} \qquad \vec{v}_2(t) = e^{(-1-i)t}\begin{bmatrix}1\\ -i\end{bmatrix}$$

and noting that $\vec{v}_1$ and $\vec{v}_2$ are complex conjugates of each other. So if we use $\vec{r}(t)$ and $\vec{s}(t)$ to denote the real and imaginary parts, respectively, of $\vec{v}_1$, (we'll compute them explicitly shortly) then

$$\vec{v}_1(t) = \vec{r}(t) + i\,\vec{s}(t) \qquad \vec{v}_2(t) = \vec{r}(t) - i\,\vec{s}(t)$$

and the general solution is

$$\begin{aligned}\vec{x}(t) &= c_1\big[\vec{r}(t) + i\,\vec{s}(t)\big] + c_2\big[\vec{r}(t) - i\,\vec{s}(t)\big] = (c_1 + c_2)\,\vec{r}(t) + (ic_1 - ic_2)\,\vec{s}(t)\\ &= d_1\vec{r}(t) + d_2\vec{s}(t)\end{aligned}$$

where $d_1 = c_1 + c_2$ and $d_2 = i(c_1 - c_2)$ are arbitrary constants, just as c_1 and c_2 were arbitrary constants. Do not fall into the trap of thinking that c_1 and c_2 are real constants so that d_2 is necessarily imaginary. As pointed out in the last example c_1 and c_2 are arbitrary *complex* constants. Whenever there are real initial conditions c_1 and c_2 will be complex in a way that leads to d_1 and d_2 being real. For example, in Example IV.7, we had $c_1 = -i$ and $c_2 = i$, so that $d_1 = 0$ and $d_2 = 2$.

Now let's find $\vec{r}(t)$ and $\vec{s}(t)$ for the $v_1(t)$ above. Recall that $\vec{r}(t)$ and $\vec{s}(t)$ are the real and imaginary parts of $\vec{v}_1(t)$. As

$$\vec{v}_1(t) = e^{(-1+i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{-t}\big[\cos t + i\sin t\big]\begin{bmatrix}1\\ i\end{bmatrix} = e^{-t}\begin{bmatrix}\cos t + i\sin t\\ i\cos t - \sin t\end{bmatrix} = e^{-t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + ie^{-t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

we have that

$$\vec{r}(t) = e^{-t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} \qquad \vec{s}(t) = e^{-t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

and the general solution is

$$\vec{x}(t) = d_1e^{-t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + d_2e^{-t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

Exercises for §IV.5

1) Find a function $\vec{x}(t)$ that obeys

a)

$$\begin{aligned} x_1'(t) &= 3x_2(t), \quad x_1(0) = 2 \\ x_2'(t) &= 3x_1(t), \quad x_2(0) = 0 \end{aligned}$$

b)

$$\begin{aligned} x_1'(t) &= -2x_1(t) - \ \ 8x_2(t), \quad x_1(0) = 4 \\ x_2'(t) &= \ \ \ 4x_1(t) + 10x_2(t), \quad x_2(0) = -1 \end{aligned}$$

c)

$$\begin{aligned} x_1'(t) &= -x_2(t) + \ \ x_3(t), \quad x_1(0) = 5 \\ x_2'(t) &= \ \ \ x_1(t) + 2x_3(t), \quad x_2(0) = -6 \\ x_3'(t) &= \ 2x_1(t) + 2x_3(t), \quad x_3(0) = -7 \end{aligned}$$

§IV.6 The Pendulum – Solution

Consider the pendulum of §IV.2 with $\frac{g}{\ell} = 2$ and $\frac{\beta}{m} = 2$. The angle θ and angular speed s then obey $\frac{d\theta}{dt} = s$, $\frac{ds}{dt} = -2\theta - 2s$ or

$$\frac{d}{dt}\begin{bmatrix}\theta\\ s\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}\theta\\ s\end{bmatrix} \qquad \text{or} \qquad \frac{d\vec{x}}{dt} = A\vec{x}$$

with

$$\vec{x} = \begin{bmatrix}\theta\\ s\end{bmatrix} \qquad\qquad A = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}$$

Try $\vec{x}(t) = e^{\lambda t}\vec{v}$ with the constants λ and $\vec{v}$ to be determined. This guess is a solution if and only if

$$\lambda e^{\lambda t}\vec{v} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix} e^{\lambda t}\vec{v}$$

Dividing both side of the equation by $e^{\lambda t}$ gives

$$\lambda\vec{v} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\vec{v} \qquad \text{or} \qquad \left(\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\vec{v} = \vec{0} \qquad \text{or} \qquad \begin{bmatrix}-\lambda & 1\\ -2 & -2-\lambda\end{bmatrix}\vec{v} = \vec{0}$$

This system of equations always has the trivial solution $\vec{v} = \vec{0}$. It has a solution with $\vec{v} \neq \vec{0}$ if and only if

$$\det\begin{bmatrix}-\lambda & 1\\ -2 & -2-\lambda\end{bmatrix} = 0$$

Evaluating the determinant

$$(-\lambda)(-2-\lambda)-(1)(-2) = \lambda^2+2\lambda+2 = (\lambda+1)^2+1 = 0 \iff (\lambda+1)^2 = -1 \iff \lambda+1 = \pm i \iff \lambda = -1\pm i$$

so the eigenvalues of A are $-1+i$ and $-1-i$.

We next find all eigenvectors of eigenvalue $-1+i$. To do so we must solve

$$\begin{bmatrix}-\lambda & 1\\ -2 & -2-\lambda\end{bmatrix}_{\lambda=-1+i}\vec{v} = \vec{0} \quad \text{or} \quad \begin{bmatrix}-(-1+i) & 1\\ -2 & -2-(-1+i)\end{bmatrix}\vec{v} = \vec{0} \quad \text{or} \quad \begin{bmatrix}1-i & 1\\ -2 & -1-i\end{bmatrix}\vec{v} = \vec{0}$$

If we not made any mechanical errors, the second row must be a multiple of the first, despite the i's floating around. Apply Gaussian elimination as usual

$$\begin{array}{c}(1)\\ (2)-\frac{-2}{1-i}(1)\end{array}\begin{bmatrix}1-i & 1\\ -2-\frac{-2}{1-i}(1-i) & -1-i-\frac{-2}{1-i}\end{bmatrix}\vec{v}=\vec{0} \quad\Longrightarrow\quad \begin{bmatrix}1-i & 1\\ 0 & -1-i-\frac{-2}{1-i}\end{bmatrix}\vec{v}=\vec{0}$$

The secret to simplifying fractions like $\frac{-2}{1-i}$ is to multiply both numerator and denominator by the complex conjugate of the denominator (which is obtained by replacing every i in the denominator by $-i$). The new denominator will be a real number.

$$\begin{bmatrix}1-i & 1\\ 0 & -1-i-\frac{-2}{1-i}\frac{1+i}{1+i}\end{bmatrix}\vec{v}=\vec{0} \quad\Longrightarrow\quad \begin{bmatrix}1-i & 1\\ 0 & -1-i-\frac{-2(1+i)}{1^2-i^2}\end{bmatrix}\vec{v}=\vec{0}$$

$$\Longrightarrow\quad \begin{bmatrix}1-i & 1\\ 0 & -1-i-\frac{-2(1+i)}{2}\end{bmatrix}\vec{v}=\vec{0} \quad\Longrightarrow\quad \begin{bmatrix}1-i & 1\\ 0 & 0\end{bmatrix}\vec{v}=\vec{0}$$

and the second row vanishes as expected. Backsolving

$$\begin{array}{l}v_2=\gamma,\ \text{arbitrary}\\ v_1=-\frac{1}{1-i}v_2=-\frac{1+i}{(1-i)(1+i)}\gamma=-\frac{1+i}{2}\gamma\end{array} \qquad \vec{v}=\frac{\gamma}{2}\begin{bmatrix}-1-i\\ 2\end{bmatrix}$$

If we have not made any mechanical errors, $\left[\begin{smallmatrix}-1-i\\ 2\end{smallmatrix}\right]$ should be an eigenvector of eigenvalue $-1+i$ (choosing $\gamma=2$ avoids fractions). That is, we should have

$$\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1-i\\ 2\end{bmatrix}=(-1+i)\begin{bmatrix}-1-i\\ 2\end{bmatrix}$$

The left and right hand sides are both equal to $\left[\begin{smallmatrix}2\\ -2+2i\end{smallmatrix}\right]$.

We could now repeat the whole computation with $\lambda=-1-i$. As we have seen before, there's an easier way. Replace every i in

$$\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1-i\\ 2\end{bmatrix}=(-1+i)\begin{bmatrix}-1-i\\ 2\end{bmatrix}$$

by $-i$. That is, take the complex conjugate.

$$\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}-1+i\\ 2\end{bmatrix}=(-1-i)\begin{bmatrix}-1+i\\ 2\end{bmatrix}$$

This is a true equation $\Big(\text{both sides equal }\left[\begin{smallmatrix}2\\ -2-2i\end{smallmatrix}\right]\Big)$ and says that $\left[\begin{smallmatrix}-1+i\\ 2\end{smallmatrix}\right]$ is an eigenvector of eigenvalue $-1-i$.

We started this example looking for solutions of $\frac{d\vec{x}}{dt}=A\vec{x}(t)$ of the form $\vec{x}(t)=e^{\lambda t}\vec{v}$. We have found (again choosing $\gamma=2$ so as to avoid fractions) that

$$\lambda=-1+i \quad \vec{v}=\begin{bmatrix}-1-i\\ 2\end{bmatrix} \qquad\qquad \lambda=-1-i \quad \vec{v}=\begin{bmatrix}-1+i\\ 2\end{bmatrix}$$

both give solutions. By linearity, for any values of c_1 and c_2,

$$\vec{x}(t)=c_1e^{(-1+i)t}\begin{bmatrix}-1-i\\ 2\end{bmatrix}+c_2e^{(-1-i)t}\begin{bmatrix}-1+i\\ 2\end{bmatrix} \quad\Longrightarrow\quad \frac{d\vec{x}}{dt}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\vec{x}(t)$$

We shall see later that there no other solutions.

Note that the solutions involve exponentials with complex exponents. The definition and main properties of such exponentials are given in Appendix C. If you are not familiar with complex exponentials, read Appendix C now. Many people avoid (foolishly – because they can substantially simplify many formulae and computations) dealing with complex exponentials. For the benefit of such people, you can always convert them into sin's and cos's, by using

$$\begin{aligned} e^{it} &= \cos t + i\sin t \\ e^{-it} &= \cos t - i\sin t \end{aligned}$$

We could just substitute in

$$\begin{aligned} (-1-i)e^{(-1+i)t} &= (-1-i)e^{-t}e^{it} &&= (-1-i)e^{-t}\big[\cos t + i\sin t\big] &&= e^{-t}\big[(-1-i)\cos t + (1-i)\sin t\big] \\ 2e^{(-1+i)t} &= 2e^{-t}e^{it} &&= 2e^{-t}\big[\cos t + i\sin t\big] &&= e^{-t}\big[2\cos t + 2i\sin t\big] \\ (-1+i)e^{(-1-i)t} &= (-1+i)e^{-t}e^{-it} &&= (-1+i)e^{-t}\big[\cos t - i\sin t\big] &&= e^{-t}\big[(-1+i)\cos t + (1+i)\sin t\big] \\ 2e^{(-1-i)t} &= 2e^{-t}e^{-it} &&= 2e^{-t}\big[\cos t - i\sin t\big] &&= e^{-t}\big[2\cos t - 2i\sin t\big] \end{aligned}$$

and collect up terms. But, by thinking a bit before computing, we can save ourselves some work.

In this application, as in most applications, the matrix A contains only real entries. So, just by taking complex conjugates of both sides of $A\vec{v} = \lambda\vec{v}$ (recall that you take complex conjugates by replacing every i with $-i$), we see that, if $\vec{v}$ is an eigenvector of eigenvalue λ, then the complex conjugate of $\vec{v}$ is an eigenvector of eigenvalue $\bar{\lambda}$. This is exactly what happened in this example. Our two eigenvalues $-1+i$, $-1-i$ are complex conjugates of each other, as are the corresponding eigenvectors $\left[\begin{smallmatrix}-1-i\\2\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}-1+i\\2\end{smallmatrix}\right]$. So our two solutions

$$\vec{x}_+(t) = e^{(-1+i)t}\begin{bmatrix}-1-i\\2\end{bmatrix} \qquad \text{and} \qquad \vec{x}_-(t) = e^{(-1-i)t}\begin{bmatrix}-1+i\\2\end{bmatrix}$$

are also complex conjugates of each other. The solution with $c_1 = c_2 = \frac{1}{2}$ (which is gotten by adding the two together and dividing by two) is thus the real part of $\vec{x}_+(t)$ and is purely real. It is

$$\begin{aligned} \tfrac{1}{2}\vec{x}_+(t) + \tfrac{1}{2}\vec{x}_-(t) &= \tfrac{1}{2}e^{(-1+i)t}\begin{bmatrix}-1-i\\2\end{bmatrix} + \tfrac{1}{2}e^{(-1-i)t}\begin{bmatrix}-1+i\\2\end{bmatrix} = \tfrac{1}{2}e^{-t}\begin{bmatrix}-e^{it} - ie^{it} - e^{-it} + ie^{-it}\\ 2e^{it} + 2e^{-it}\end{bmatrix} \\ &= e^{-t}\begin{bmatrix}-\cos t + \sin t\\ 2\cos t\end{bmatrix} \end{aligned}$$

In the last step we used

$$\begin{aligned} e^{it} + e^{-it} &= 2\cos t \\ e^{it} - e^{-it} &= 2i\sin t \end{aligned}$$

The solution with $c_1 = \frac{1}{2i}$ and $c_2 = -\frac{1}{2i}$ (which is gotten by subtracting the second solution from the first and dividing by $2i$) is the imaginary part of $\vec{x}_+(t)$ and is also purely real. It is

$$\begin{aligned} \tfrac{1}{2i}\vec{x}_+(t) - \tfrac{1}{2i}\vec{x}_-(t) &= \tfrac{1}{2i}e^{(-1+i)t}\begin{bmatrix}-1-i\\2\end{bmatrix} - \tfrac{1}{2i}e^{(-1-i)t}\begin{bmatrix}-1+i\\2\end{bmatrix} = \tfrac{1}{2i}e^{-t}\begin{bmatrix}-e^{it} - ie^{it} + e^{-it} - ie^{-it}\\ 2e^{it} - 2e^{-it}\end{bmatrix} \\ &= e^{-t}\begin{bmatrix}-\sin t - \cos t\\ 2\sin t\end{bmatrix} \end{aligned}$$

We now have two purely real solutions. By linearity, any linear combination of them is also a solution. So, for any values of a and b

$$\vec{x}(t) = ae^{-t}\begin{bmatrix}-\cos t + \sin t\\ 2\cos t\end{bmatrix} + be^{-t}\begin{bmatrix}-\sin t - \cos t\\ 2\sin t\end{bmatrix} \quad \Longrightarrow \quad \frac{d\vec{x}}{dt} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\vec{x}(t)$$

We have not constructed any solutions that we did not have before. We have really just renamed the arbitrary constants. To see this, just substitute back

$$\begin{aligned} \vec{x}(t) &= ae^{-t}\begin{bmatrix}-\cos t + \sin t\\ 2\cos t\end{bmatrix} + be^{-t}\begin{bmatrix}-\sin t - \cos t\\ 2\sin t\end{bmatrix} \\ &= \tfrac{a}{2}\big(\vec{x}_+(t) + \vec{x}_-(t)\big) + \tfrac{b}{2i}\big(\vec{x}_+(t) - \vec{x}_-(t)\big) = \big(\tfrac{a}{2} + \tfrac{b}{2i}\big)x_+(t) + \big(\tfrac{a}{2} - \tfrac{b}{2i}\big)x_-(t) \\ &= c_1\vec{x}_+(t) + c_2\vec{x}_-(t) \end{aligned}$$

with $c_1 = \frac{a}{2} + \frac{b}{2i}$, $c_2 = \frac{a}{2} - \frac{b}{2i}$. In most applications, a and b turn out to be real and c_1 and c_2 turn out to be complex.

Exercises for §IV.6

1) Find all eigenvalues and eigenvectors of each of the following matrices.

$$a)\ \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \qquad b)\ \begin{bmatrix}0 & -1\\ 5 & 2\end{bmatrix} \qquad c)\ \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

2) Find a function $\vec{x}(t)$ that obeys

a)

$$\begin{aligned} x_1'(t) &= x_1(t) - x_2(t), \quad & x_1(0) = 1\\ x_2'(t) &= x_1(t) + x_2(t), \quad & x_2(0) = 1 \end{aligned}$$

b)

$$\begin{aligned} x_1'(t) &= -x_2(t), \quad & x_1(0) = 1\\ x_2'(t) &= 5x_1(t) + 2x_2(t), \quad & x_2(0) = 1 \end{aligned}$$

§IV.7 Matrix Powers

Suppose that we are interested in the long time behaviour of some random walk problem, as in §III.3. In such a problem, we are given a time zero configuration $\vec{x}_0$ and a transition matrix P. At time n, the configuration is $\vec{x}_n = P^n\vec{x}_0$. So to determine the behaviour of $\vec{x}_n$ for very large n, we need to be able to compute the large power P^n of P. Computing P^n by repeated matrix multiplication is extremely demanding. It is far easier to use eigenvalues and eigenvectors and the following observations:

(a) If $\vec{v}$ is an eigenvector of the matrix A with eigenvalue λ, then $A^n\vec{v} = \lambda^n\vec{v}$. Here is how to see that that is the case.

$$\begin{aligned} A\vec{v} &= \lambda\vec{v}\\ A^2\vec{v} &= A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}\\ A^3\vec{v} &= A(A^2\vec{v}) = A(\lambda^2\vec{v}) = \lambda^2 A\vec{v} = \lambda^2(\lambda\vec{v}) = \lambda^3\vec{v}\\ &\ \vdots \end{aligned}$$

(b) If B is any $m\times m$ matrix, then $B = \big[B\hat{\mathbf{e}}_1\ B\hat{\mathbf{e}}_2\ \cdots\ B\hat{\mathbf{e}}_m\big]$. For example

$$B = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix} \implies B\hat{\mathbf{e}}_2 = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}b\\e\\h\end{bmatrix} = 2^{\text{nd}}\text{ column of } B$$

Observation (b) says that, to find A^n, it sufices to find each column vector $A^n\hat{\mathbf{e}}_j$. If we can express $\hat{\mathbf{e}}_j$ as a linear combination of eigenvectors, then we can use observation (a) to compute $A^n\hat{\mathbf{e}}_j$. Here is am example.

Example IV.9 Let's compute A^{10} for the matrix

$$A = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$$

We saw, in Example IV.3, that $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of eigenvalue -1 and that $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector of eigenvalue -3. Consequently

$$A^{10}\begin{bmatrix}1\\1\end{bmatrix} = (-1)^{10}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \qquad A^{10}\begin{bmatrix}1\\-1\end{bmatrix} = (-3)^{10}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}59049\\-59049\end{bmatrix}$$

Since

$$\widehat{\mathbf{e}}_1 = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} \qquad \widehat{\mathbf{e}}_1 = \begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

we have

$$A^{10}\widehat{\mathbf{e}}_1 = \frac{1}{2}A^{10}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}A^{10}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}59049\\-59049\end{bmatrix} = \begin{bmatrix}29525\\-29524\end{bmatrix}$$

$$A^{10}\widehat{\mathbf{e}}_2 = \frac{1}{2}A^{10}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}A^{10}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}59049\\-59049\end{bmatrix} = \begin{bmatrix}-29524\\29525\end{bmatrix}$$

and hence

$$A^{10} = \big[A^{10}\widehat{\mathbf{e}}_1\ A^{10}\widehat{\mathbf{e}}_2\big] = \begin{bmatrix}29525 & -29524\\-29524 & 29525\end{bmatrix}$$

Now, let's return to studying the long time behaviour of random walks. Let P be any $m\times m$ transition matrix. It turns out that

- 1 is always an eigenvalue of P.
- Every eigenvalue λ of P obeys $|\lambda|\le 1$. Usually (but not always), P has eigenvalue 1 with mutliplicity one and all other eigenvalues of P obey $|\lambda|<1$.
- Usually (but not always) every vector $\vec x_0$ can be written as a linear combination of eigenvectors.

Suppose that $\vec v_1,\ \vec v_2,\ \cdots,\ \vec v_m$ are eigenvectors of P with eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_m$ respectively. Suppose further that $\lambda_1=1$ and $|\lambda_j|<1$ for all $j\ge 2$ and that $\vec x_0 = c_1\vec v_1 + c_2\vec v_2+\cdots+c_m\vec v_m$. Then

$$\vec x_n = P^n\vec x_0 = c_1P^n\vec v_1 + c_2P^n\vec v_2+\cdots+c_mP^n\vec v_m = c_1\lambda_1^n\vec v_1 + c_2\lambda_2^n\vec v_2+\cdots+c_m\lambda_m^n\vec v_m$$

Now $\lambda_1^n = 1^n = 1$ for all n. But for $j\ge 2$, $\lambda_j^n\to 0$ as $n\to\infty$, since $|\lambda_j|<1$. Consequently

$$\lim_{n\to\infty}\vec x_n = c_1\vec v_1$$

The constant c_1 is determined by the requirement that the sum of the components of every $\vec x_n$, and hence of $\lim\limits_{n\to\infty}\vec x_n$, must be exactly 1. Consequently, we always have the same limit $\lim\limits_{n\to\infty}\vec x_n = c_1\vec v_1$, regardless of what the initial configuration $\vec x_0$ was. This limit is called the equilibrium or steady state of the random walk.

§IV.8 Diagonalization

Recall that

$$\begin{aligned}\lambda \text{ is an eigenvalue of } A &\iff (A-\lambda I)\vec v = \vec 0 \text{ has a nonzero solution}\\ &\iff \det(A-\lambda I)=0\\ &\iff \lambda \text{ is a root of } C_A(\lambda)=\det(A-\lambda I)\end{aligned}$$

and that $C_A(\lambda)$ is called the characteristic polynomial of A. As its name suggests, the characteristic polynomial of a matrix is always a polynomial. For a general 2×2 matrix,

$$\begin{aligned}A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \quad\Longrightarrow\quad C_A(\lambda) &= \det\begin{bmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{bmatrix} = \big(a_{11}-\lambda\big)\big(a_{22}-\lambda\big) - a_{12}a_{21}\\ &= (-\lambda)^2 - \big(a_{11}+a_{22}\big)\lambda - a_{12}a_{21}\end{aligned}$$

the characteristic polynomial is always a polynomial of degree two whose term of degree two is always $(-\lambda)^2$. For a general 3×3 matrix

$$\begin{aligned}
C_A(\lambda) &= \det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33-\lambda}\end{bmatrix} \\
&= \big(a_{11}-\lambda\big)\det\begin{bmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda\end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda\end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22}-\lambda \\ a_{31} & a_{32}\end{bmatrix} \\
&= \big(a_{11}-\lambda\big)\det\begin{bmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda\end{bmatrix} + \text{ a polynomial of degree 1 in } \lambda \\
&= \big(a_{11}-\lambda\big)\big(a_{22}-\lambda\big)\big(a_{33}-\lambda\big) + \text{ a polynomial of degree 1 in } \lambda \\
&= (-\lambda)^3 + \text{ a polynomial of degree 2 in } \lambda
\end{aligned}$$

For an $n\times n$ matrix, expanding as we did for a 3×3 matrix, yields that

$$C_A(\lambda) = (-\lambda)^n + \text{ a polynomial of degree } n-1 \text{ in } \lambda$$

The $(-\lambda)^n$ comes from multiplying out the product $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$ of diagonal entries. The power of λ in all other terms is strictly smaller than n. Every such polynomial can be written in the form

$$C_A(\lambda) = (-1)^n(\lambda-\lambda_1)^{p_1}\ \cdots\ (\lambda-\lambda_k)^{p_k}$$

where each λ_i is a real or complex number and each p_i is an integer obeying $p_i\ge 1$, which is called the multiplicity of λ_i. Every polynomial of degree n has precisely n roots, counting multiplicity and every $n\times n$ matrix has precisely n eigenvalues, counting multiplicity. There is an appendix to this chapter giving some tricks that help you find roots of polynomials.

For each *different* eigenvalue λ_j we are guaranteed the existence of a corresponding eigenvector, because

$$\det(A-\lambda_j I) = 0 \quad\Longrightarrow\quad (A-\lambda_j I)\vec v = \vec 0 \text{ has a nontrivial solution}$$

Let $\vec v_j$ be an eigenvector of A of eigenvalue λ_j. We are now going to derive a formula for A that combines the n equations

$$A\vec v_j = \lambda_j\vec v_j \qquad j = 1,\ 2,\ \cdots,\ n$$

into one big equation. We'll first do a specific example and then derive the formula in general.

Example IV.10 The eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1\end{bmatrix}$$

are

$$\lambda_1 = 4 \quad \vec v_1 = \begin{bmatrix} 1 \\ 1\end{bmatrix} \qquad\qquad \lambda_1 = -2 \quad \vec v_1 = \begin{bmatrix} 1 \\ -1\end{bmatrix}$$

That is,

$$A\vec v_1 = \begin{bmatrix} 1 & 3 \\ 3 & 1\end{bmatrix}\begin{bmatrix} 1 \\ 1\end{bmatrix} = \begin{bmatrix} 1\times 1+3\times 1 \\ 3\times 1+1\times 1\end{bmatrix} = 4\begin{bmatrix} 1 \\ 1\end{bmatrix} \quad A\vec v_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1\end{bmatrix}\begin{bmatrix} 1 \\ -1\end{bmatrix} = \begin{bmatrix} 1\times 1+3\times(-1) \\ 3\times 1+1\times(-1)\end{bmatrix} = -2\begin{bmatrix} 1 \\ -1\end{bmatrix}$$

Let

$$U = [\vec v_1, \vec v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}$$

be the matrix whose $j^{\rm th}$ column is the $j^{\rm th}$ eigenvector $\vec v_j$. Note that the $j^{\rm th}$ column of the product AU is, by the usual rules of matrix multiplication, A times the $j^{\rm th}$ column of U, which is $A\vec v_j = \lambda_j \vec v_j$.

$$AU = \begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}1\times1+3\times1 & 1\times1+3\times(-1)\\ 3\times1+1\times1 & 3\times1+1\times(-1)\end{bmatrix} = [A\vec v_1\ A\vec v_2] = \begin{bmatrix}4\times1 & -2\times1\\ 4\times1 & -2\times(-1)\end{bmatrix}$$

If it weren't for the eigenvalues 4, -2 multiplying the two columns of the final matrix, the final matrix would just be U once again. These two eigenvalues can be "pulled out" out the final matrix

$$\begin{bmatrix}4\times1 & -2\times1\\ 4\times1 & -2\times(-1)\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & -2\end{bmatrix}$$

All together

$$\begin{bmatrix}1 & 3\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & -2\end{bmatrix} \qquad \text{or} \qquad AU = UD$$

where the columns of U are the eigenvectors of A and D is a diagonal matrix having the eigenvalues of A running down the diagonal.

Now back to the general case. Let $U = [\vec v_1 \ \cdots\ \vec v_n]$ be the $n\times n$ matrix whose $j^{\rm th}$ column is the $j^{\rm th}$ eigenvector $\vec v_j$. Then

$$\begin{aligned}
& A\vec v_j = \lambda_j \vec v_j \qquad\qquad \text{for } j=1,\cdots,n\\
\implies\ & A[\vec v_1\ \cdots\ \vec v_n] = [A\vec v_1\ \cdots\ A\vec v_n] = [\lambda_1\vec v_1\ \cdots\ \lambda_n\vec v_n]\\
\implies\ & A[\vec v_1\ \cdots\ \vec v_n] = [\vec v_1\ \cdots\ \vec v_n]\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}\\
\implies\ & AU = UD
\end{aligned}$$

where D is a diagonal matrix whose (j,j) matrix element is the $j^{\rm th}$ eigenvalue. (WARNING: Be careful that the eigenvectors and eigenvalues are placed in the same order.) In the event that U is invertible

$$A = UDU^{-1} \qquad\qquad D = U^{-1}AU$$

As we shall see in the next section, diagonal matrices are easy to compute with and matrices of the form UDU^{-1} with D diagonal are almost as easy to compute with. Matrices that can be written in the form $A = UDU^{-1}$, with D diagonal, are called diagonalizable. Not all matrices are diagonalizable. For example

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

which has eigenvalues $\lambda = 0, 0$ and eigenvectors $c\begin{bmatrix}1\\0\end{bmatrix}$, $c\neq 0$ is not diagonalizable. But, in practice, most matrices that you will encounter will be diagonalizable. In particular every $n\times n$ matrix that obeys at least one of the following conditions

- A has no multiple eigenvalues
- $A_{ij} = \overline{A}_{ji}$, for all $1\le i,j\le n$ (such matrices are called self–adjoint or hermitian)
- A_{ij} is real and $A_{ij} = A_{ji}$, for all $1\le i,j\le n$ (such matrices are called real, symmetric)

is diagonalizable.

Exercises for §IV.8

1) Show that

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

cannot be written in the form UDU^{-1} with U an invertible 2×2 matrix and D a diagonal 2×2 matrix.

2) Set $A = \frac{1}{5}\begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}$.

a) Evaluate A^{10}.

b) Evaluate $\lim\limits_{n\to\infty} \frac{A^n\vec v}{\|A^n\vec v\|}$ for all vectors $\vec v \ne \vec 0$.

3) Find **all** square roots of $A = \frac{1}{5}\begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix}$. That is, find all matrices B obeying $B^2 = A$.

§IV.9 The General Solution of $\frac{d\vec x}{dt} = A\vec x$.

We have seen how to guess many solutions of $\frac{d\vec x}{dt} = A\vec x$. In order for guessing to be an efficient method for finding solutions, you have to know when to stop guessing. You have to be able to determine when you have found all solutions. There is one system in which it is easy to determine the general solution. When $A = 0$, the system reduces to

$$\frac{d\vec x}{dt} = \vec 0$$

So every component x_i is independent of time and the general solution is

$$\vec x(t) = \vec c$$

where $\vec c$ is a vector of arbitrary constants.

There is a trick which enables us, in principal, to write every system $\frac{d\vec x}{dt} = A\vec x$ in the form $\frac{d}{dt}\overrightarrow{(\text{something})} = \vec 0$. The trick uses the exponential matrix e^{-At}, which is defined by

$$e^{-At} = I + (-At) + \tfrac{1}{2}(-At)^2 + \tfrac{1}{3!}(-At)^3 + \tfrac{1}{4!}(-At)^4 + \cdots$$

The exponential obeys

$$\begin{aligned}\tfrac{d}{dt}e^{-At} &= 0 + (-A) + (-At)(-A) + \tfrac{1}{2!}(-At)^2(-A) + \tfrac{1}{3!}(-At)^3(-A) + \cdots \\ &= \big[I + (-At) + \tfrac{1}{2}(-At)^2 + \tfrac{1}{3!}(-At)^3 + \cdots\big](-A) \\ &= -e^{-At}A = -Ae^{-At}\end{aligned}$$

just as if A were a number. If we multiply both sides of

$$\tfrac{d\vec x}{dt}(t) - A\vec x(t) = \vec 0$$

by e^{-At}, the left hand side

$$e^{-At}\tfrac{d\vec x}{dt}(t) - e^{-At}A\vec x(t) = \tfrac{d}{dt}\left(e^{-At}\vec x(t)\right)$$

is a perfect derivative. So $\frac{d\vec x}{dt}(t) = A\vec x(t)$ is equivalent to $\frac{d}{dt}\left(e^{-At}\vec x(t)\right) = 0$, which has general solution

$$e^{-At}\vec x(t) = \vec c \qquad \text{or} \qquad \vec x(t) = e^{At}\vec c$$

If A is an $n\times n$ matrix, there are n arbitrary constants in the general solution.

We have now transformed the problem of solving $\frac{d\vec x}{dt} = A\vec x$ into the problem of computing e^{At}. When A is diagonalizable (and, in practice, A almost always is diagonalizable) it is easy to compute e^{At}. Suppose that $A = UDU^{-1}$ with

$$U = [\vec v_1 \ \cdots \ \vec v_n] \qquad\qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

It is trivial to compute the exponential of a diagonal matrix because

$$D^2 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^3 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^3 \end{bmatrix}$$

$$\vdots$$

$$D^k = D^{k-1} D = \begin{bmatrix} \lambda_1^{k-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{k-1} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

Every power of a diagonal matrix is gotten by just taking the corresponding powers of the matrix elements on the diagonal. So, for any function $f(x)$ that is given by the sum of a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$$f(D) = \sum_{k=0}^{\infty} a_k D^k = \sum_{k=0}^{\infty} a_k \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} a_k \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{n=0}^{\infty} a_k \lambda_n^k \end{bmatrix} = \begin{bmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{bmatrix}$$

In particular,

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

For a diagonalizable (though not necessarily diagonal) matrix A,

$$\begin{aligned} A^2 = AA &= UDU^{-1}UDU^{-1} &&= UDIDU^{-1} &&= UD^2U^{-1} \\ A^3 = A^2A &= UD^2U^{-1}UDU^{-1} &&= UD^2IDU^{-1} &&= UD^3U^{-1} \\ &\vdots \\ A^k = A^{k-1}A &= UD^{k-1}U^{-1}UDU^{-1} &&= UD^{k-1}IDU^{-1} &&= UD^kU^{-1} \end{aligned}$$

So, for any function $f(x)$ that is given by the sum of a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$$f(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k=0}^{\infty} a_k U D^k U^{-1} = U f(D) U^{-1}$$

In particular,

$$e^{At} = Ue^{Dt}U^{-1} = U \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}$$

We now return to the problem of evaluating the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$, when A is diagonalizable with eigenvalues $\lambda_1, \cdots, \lambda_n$ and corresponding eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$. The general solution is

$$e^{At}\vec{c} = U \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} U^{-1}\vec{c} = [\vec{v}_1 \ \cdots \ \vec{v}_n] \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \vec{d}$$

where $\vec{d} = U^{-1}\vec{c}$ is a new vector of arbitrary constants. Multiplying everything out

$$e^{At}\vec{c} = [\vec{v}_1 \ \cdots \ \vec{v}_n] \begin{bmatrix} e^{\lambda_1 t} d_1 \\ \vdots \\ e^{\lambda_n t} d_n \end{bmatrix} = d_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + d_n e^{\lambda_n t}\vec{v}_n$$

we conclude that the general solution is a linear combination of n terms (recall that A is $n \times n$) with the $j^{\rm th}$ term being an arbitrary constant times $e^{\lambda_j t}\vec{v}_j$ where λ_j and $\vec{v}_j$ are the $j^{\rm th}$ eigenvalue and eigenvector, respectively. This is precisely the form that we found in §IV.5 and §IV.6.

Example IV.11 As we saw in example IV.10, the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

are

$$\lambda_1 = 4 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \lambda_1 = -2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence A is diagonalizable and

$$A = UDU^{-1} \qquad \text{with} \qquad U = [\vec{v}_1, \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$$

The exponential of a diagonal matrix is obtained by exponentiating the diagonal entries

$$e^{Dt} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

and the exponential of At is obtained by sandwiching e^{Dt} between U and U^{-1}

$$e^{At} = Ue^{Dt}U^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

The general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ is

$$\begin{aligned} \vec{x}(t) = e^{At}\vec{c} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} d_1 \\ e^{-2t} d_2 \end{bmatrix} = \begin{bmatrix} e^{4t} d_1 + e^{-2t} d_2 \\ e^{4t} d_1 - e^{-2t} d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} d_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} d_2 \\ &= d_1 e^{\lambda_1 t}\vec{v}_1 + d_2 e^{\lambda_2 t}\vec{v}_2 \end{aligned}$$

as expected. Here $\vec{c}$ and $\vec{d}$ are both vectors of arbitrary constants with $\vec{d} = U^{-1}\vec{c}$.

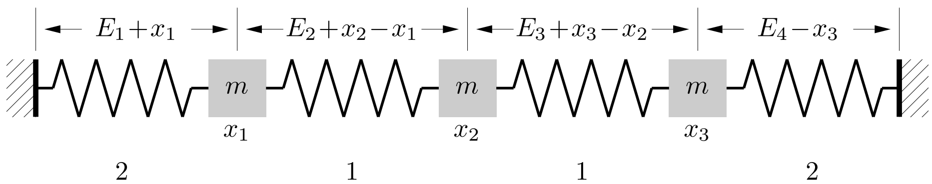
Exercises for §IV.9

1) Evaluate e^{At} for

a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

§IV.10 Coupled Springs

Consider the following system of three masses coupled by springs



position w.r.t equilibrium
spring constant

To make the numbers work out nicely, I have chosen the three masses equal (say to m) and the four spring constants to be, in some units, 2, 1, 1, 2 respectively. Furthermore, we use a coordinate system in which x_i is the position of mass number i, relative to its equilibrium position. So, when the system is in equilibrium $x_1 = x_2 = x_3 = 0$. Let us denote by $E_j,\ j = 1, 2, 3, 4$ the lengths of the four springs at equilibrium and by $\ell_j,\ j = 1, 2, 3, 4$ the natural lengths of the four springs. Then the length at time t of the spring joining the masses at x_2 and x_3 is $x_3(t) - x_2(t) + E_3$. The force exerted by an ideal spring is, up to a sign, the spring constant of the spring times (its length minus its natural length). Newton's law of motion for this system (assuming that there are no frictional forces, that the springs are massless and that the masses are constrained to move horizontally) is thus

$$\begin{aligned}
mx_1'' &= -2(x_1 + E_1 - \ell_1) + (x_2 - x_1 + E_2 - \ell_2)\\
mx_2'' &= -(x_2 - x_1 + E_2 - \ell_2) + (x_3 - x_2 + E_3 - \ell_3)\\
mx_3'' &= -(x_3 - x_2 + E_3 - \ell_3) + 2(-x_3 + E_4 - \ell_4)
\end{aligned}$$

The easy way to check that the signs are correct here is to pretend that $x_3 \gg x_2 \gg x_1 \gg 0$ so that all springs except the last are very stretched out. For example, the second spring tries to pull the x_1 mass to the right (which is consistent with the $+(x_2 - x_1 + E_2 - \ell_2) > 0$ force term in the first equation) and the x_2 mass to the left (which is consistent with the $-(x_2 - x_1 + E_2 - \ell_2) < 0$ force term in the second equation).

When the system is at rest at equilibrium $x_1 = x_2 = x_3 = x_1'' = x_2'' = x_3'' = 0$ and the equations of motion simplify to

$$\begin{aligned}
0 &= -2(E_1 - \ell_1) + (E_2 - \ell_2)\\
0 &= -(E_2 - \ell_2) + (E_3 - \ell_3)\\
0 &= -(E_3 - \ell_3) + 2(E_4 - \ell_4)
\end{aligned}$$

so the equilibrium lengths obey $2(E_1 - \ell_1) = E_2 - \ell_2 = E_3 - \ell_3 = 2(E_4 - \ell_4)$. Substituting these into the full equations of motion causes all of the E_j's and ℓ_j's to cancel out giving

$$\begin{aligned}
mx_1'' &= -2x_1 + (x_2 - x_1) &&= -3x_1 + x_2\\
mx_2'' &= -(x_2 - x_1) + (x_3 - x_2) &&= x_1 - 2x_2 + x_3\\
mx_3'' &= -(x_3 - x_2) - 2x_3 &&= x_2 - 3x_3
\end{aligned}$$

or $m\vec{x}'' = F\vec{x}$ with

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad F = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

We could convert this from a second order system into a first order system, but it is not necessary. Let's look for solutions of the form $\vec{x}(t) = e^{\mu t}\vec{v}$ with the constants μ and $\vec{v}$ to be determined. Substituting,

$$\vec{x}(t) = e^{\mu t}\vec{v} \text{ is a solution } \iff m\mu^2 e^{\mu t}\vec{v} = Fe^{\mu t}\vec{v} \quad \text{for all } t \iff m\mu^2\vec{v} = F\vec{v}$$

we see that the guess is a nontrivial solution if and only $\vec{v}$ is an eigenvector of F of eigenvalue $m\mu^2$.

We have to find the eigenvalues and eigenvectors of F. Expand $\det(F-\lambda I)$ along its first row. Remember that we wish to find the roots of this polynomial. So we try to express $\det(F-\lambda I)$ as a product of factors, rather than as a polynomial in standard form.

$$
\begin{aligned}
0 &= \det\begin{bmatrix} -3-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -3-\lambda \end{bmatrix} \\
&= \big(-3-\lambda\big)\det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} - \det\begin{bmatrix} 1 & 1 \\ 0 & -3-\lambda \end{bmatrix} \\
&= -\big(3+\lambda\big)\big[(2+\lambda)\big(3+\lambda\big)-1\big] - \big(-3-\lambda\big) \\
&= -\big(3+\lambda\big)\big[\lambda^2+5\lambda+6-1-1\big] \\
&= -\big(3+\lambda\big)\big[\lambda^2+5\lambda+4\big] \\
&= -\big(3+\lambda\big)\big(\lambda+4\big)\big(\lambda+1\big)
\end{aligned}
$$

The eigenvalues are $\lambda = -1,\ -3,\ -4$. To find the corresponding eigenvectors, we must solve $(F-\lambda I)\vec{x} = \vec{0}$ for $\vec{x}$.

$\lambda = -1:$

$$
\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]
\quad
\begin{array}{c} (1) \\ 2(2)+(1) \\ (3) \end{array}
\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]
\quad
\begin{array}{c} (1) \\ (2) \\ (3)+(2) \end{array}
\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\implies \vec{x} = \alpha\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}
$$

$\lambda = -3:$

$$
\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]
\quad
\begin{array}{c} (2) \\ (1) \\ (3)-(1) \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\implies \vec{x} = \beta\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
$$

$\lambda = -4:$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]
\quad
\begin{array}{c} (1) \\ (2)-(1) \\ (3) \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]
\quad
\begin{array}{c} (1) \\ (2) \\ (3)-(2) \end{array}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\implies \vec{x} = \gamma\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
$$

For each eigenvalue λ there are two corresponding values of μ, gotten by solving $m\mu^2 = \lambda$. Let's take $m = 1$. The the values of μ that correspond to $\lambda = -1$ are the two solutions of $\mu^2 = -1$ or $\mu = \pm i$. All of the following are solutions to $m\vec{x}'' = F\vec{x}$

$$
\alpha e^{it}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}
\qquad \alpha' e^{-it}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}
\qquad \beta e^{i\sqrt{3}t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\qquad \beta' e^{-i\sqrt{3}t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\qquad \gamma e^{i2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\qquad \gamma' e^{-i2t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
$$

The general solution is

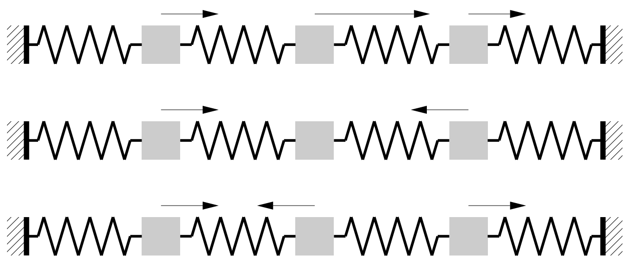
$$
\begin{aligned}
\vec{x}(t) &= \big(\alpha e^{it} + \alpha' e^{-it}\big)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \big(\beta e^{i\sqrt{3}t} + \beta' e^{-i\sqrt{3}t}\big)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \big(\gamma e^{i2t} + \gamma' e^{-i2t}\big)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= \big(a\cos t + a'\sin t\big)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \big(b\cos(\sqrt{3}t) + b'\sin(\sqrt{3}t)\big)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \big(c\cos(2t) + c'\sin(2t)\big)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\end{aligned}
$$

To convert, for example, from the apparently complex $\big(\alpha e^{it}+\alpha' e^{-it}\big)$ to the apparently real $\big(a\cos t+a'\sin t\big)$ we used

$$\alpha e^{it}+\alpha' e^{-it}=\alpha[\cos t+i\sin t]+\alpha'[\cos t-i\sin t]=(\alpha+\alpha')\cos t+i(\alpha-\alpha')\sin t$$

and renamed the arbitrary constants $\alpha+\alpha'=a,\ i(\alpha-\alpha')=a'$.

There are three "modes of oscillation", with frequencies 1, $\sqrt{3}$ and 2 radians per second. For the first mode, all three masses are always moving in the same direction, but the middle one has twice the amplitude of the two outside masses. For the second mode, the middle remains stationary and the outside two are always moving in opposite directions. For the third mode, all three masses have the same amplitude, but the middle mass always moves in the opposite direction to the outside two.



§IV.11 Worked Problems

Questions

1) Find the eigenvalues and eigenvectors of

a) $\begin{bmatrix}0 & -2\\ 1 & 3\end{bmatrix}$ b) $\begin{bmatrix}-3 & -2\\ 15 & 8\end{bmatrix}$ c) $\begin{bmatrix}3 & 2\\ -1 & 1\end{bmatrix}$

2) Find the eigenvalues and eigenvectors of

a) $\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$ b) $\begin{bmatrix}2 & 1 & 0\\ 6 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$ c) $\begin{bmatrix}2 & 0 & 6\\ 0 & 3 & 1\\ 0 & 0 & 3\end{bmatrix}$

3) Find the functions $x_1(t)$ and $x_2(t)$ satisfying

a) $\begin{aligned} x_1'(t) &= 2x_1(t)+2x_2(t) & x_1(0)&=1\\ x_2'(t) &= 2x_1(t)-\ \ x_2(t) & x_2(0)&=0\end{aligned}$

b) $\begin{aligned} x_1'(t) &= -2x_1(t)+5x_2(t) & x_1(0)&=\ \ 1\\ x_2'(t) &= \ \ \ 4x_1(t)+6x_2(t) & x_2(t)&=-1\end{aligned}$

4) Find the functions $x_1(t),\ x_2(t)$ and $x_3(t)$ satisfying

$$\begin{aligned} x_1'(t) &= 2x_1(t)+x_2(t) & x_1(0)&=1\\ x_2'(t) &= 6x_1(t)+x_2(t)-x_3(t) & x_2(0)&=2\\ x_3'(t) &= \qquad\qquad\qquad\quad\ x_3(t) & x_3(0)&=3\end{aligned}$$

5) Find the functions $x_1(t),\ x_2(t)$ and $x_3(t)$ satisfying

$$\begin{aligned} x_1'(t) &= \ \ 2x_1(t)-6x_2(t)-6x_3(t) & x_1(0)&=0\\ x_2'(t) &= -x_1(t)+\ \ x_2(t)+2x_3(t) & x_2(0)&=1\\ x_3'(t) &= \ \ 3x_1(t)-6x_2(t)-7x_3(t) & x_3(0)&=0\end{aligned}$$

6) Let

$$A=\begin{bmatrix}-3 & 0 & 2\\ 1 & -1 & 0\\ -2 & -1 & 0\end{bmatrix}$$

Find the eigenvalues and eigenvectors of A

7) Consider the differential equation $\vec{x}'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

Find (a) the general solution and (b) the solution that satisfies the initial conditions $x_1(0) = 2,\ x_2(0) = 0,\ x_3(0) = -1$.

8) State whether each of the following statments is true or false. In each case give a *brief* reason.

a) The matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has no eigenvectors.

b) The vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

c) If λ is an eigenvalue of the matrix A, then λ^3 is a eigenvalue of the matrix A^3.

d) If 0 is an eigenvalue of the matrix A, then A is not invertible.

9) Find, if possible, a matrix A obeying

$$A^3 = \begin{bmatrix} -34 & -105 \\ 14 & 43 \end{bmatrix}$$

10) Find a 3×3 matrix M having the eigenvalues 1 and 2, such that the eigenspace for $\lambda = 1$ is a line whose direction vector is $[2, 0, 1]$ and the eigenspace for $\lambda = 2$ is the plane $x - 2y + z = 0$.

11) Consider a population which is divided into three types and reproduces as follows:

70% of the offspring of type 1 are type 1, 10% are type 2 and 20% are type 3

10% of the offspring of type 2 are type 1, 80% are type 2 and 10% are type 3

10% of the offspring of type 3 are type 1, 30% are type 2 and 60% are type 3

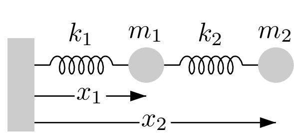
All three types reproduce at the same rate. Let $x_i(n)$ denote the fraction of generation n which is of type i, for $i = 1, 2, 3$.

a) Find a matrix A such that $\vec{x}(n+1) = A\vec{x}(n)$.

b) Find the eigenvalues of A.

c) Is there an equilibrium distribution, i.e. a vector $\vec{x}$ such that $\vec{x}(n) = \vec{x}$ for all n if $\vec{x}(0) = \vec{x}$? If so, find it.

12) Consider the following mass-spring system on a frictionless plane. Both masses are 1 kg. and the natural

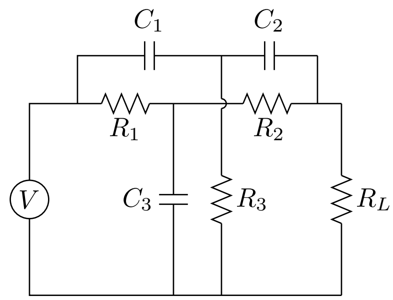


length of both springs is 1 m. Their spring constants are k_1 and k_2. Let $x_i,\ i = 1, 2$ denote the distance of mass i from the wall at the left.

a) Use Newton's law to find the equations of motion of the masses.

b) Write these equations as a first order system of differential equations.

13) The circuit below is known as a twin–T RC network and is used as a filter. Find a system of equations that determine the various currents flowing in the circuit, asssuming that the applied voltage, $V = V(t)$, is given.



Solutions

1) Find the eigenvalues and eigenvectors of

a) $\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$ b) $\begin{bmatrix} -3 & -2 \\ 15 & 8 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$

Solution. a) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\det(A - \lambda I) = \det \begin{bmatrix} 0-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} = (-\lambda)(3-\lambda) - (-2) = \lambda^2 - 3\lambda + 2 = 0$$

or $\boxed{\lambda = 1, 2}$. The eigenvectors corresponding to $\lambda = 1$ are all nonzero solutions of the linear system of equations

$$(A - I)\vec{y} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{c \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ c \neq 0}$$

and those corresponding to $\lambda = 2$ are all nonzero solutions of the linear system of equations

$$(A - 2I)\vec{y} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ c \neq 0}$$

As a check, note that

$$\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

b) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & -2 \\ 15 & 8-\lambda \end{bmatrix} = (-3-\lambda)(8-\lambda) - (-30) = \lambda^2 - 5\lambda + 6 = 0$$

or $\boxed{\lambda = 2, 3}$. The eigenvectors corresponding to $\lambda = 2$ are all nonzero solutions of the linear system of equations

$$(A - 2I)\vec{y} = \begin{bmatrix} -5 & -2 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{c \begin{bmatrix} 2 \\ -5 \end{bmatrix},\ c \neq 0}$$

and those corresponding to $\lambda = 3$ are all nonzero solutions of the linear system of equations

$$(A - 3I)\vec{y} = \begin{bmatrix} -6 & -2 \\ 15 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ -3 \end{bmatrix},\ c \neq 0}$$

As a check, note that

$$\begin{bmatrix}-3&-2\\15&8\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=2\begin{bmatrix}2\\-5\end{bmatrix}\qquad\qquad \begin{bmatrix}-3&-2\\15&8\end{bmatrix}\begin{bmatrix}1\\-3\end{bmatrix}=3\begin{bmatrix}1\\-3\end{bmatrix}$$

c) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\det(A-\lambda I)=\det\begin{bmatrix}3-\lambda&2\\-1&1-\lambda\end{bmatrix}=(3-\lambda)(1-\lambda)-(-2)=\lambda^2-4\lambda+5=0$$

or $\lambda=\frac{1}{2}(4\pm\sqrt{16-20})=\boxed{2\pm i}$. The eigenvectors corresponding to $\lambda=2+i$ are all nonzero solutions of the linear system of equations

$$(A-(2+i)I)\vec{y}=\begin{bmatrix}1-i&2\\-1&-1-i\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\quad\Longrightarrow\quad\begin{bmatrix}y_1\\y_2\end{bmatrix}=\boxed{c\begin{bmatrix}1+i\\-1\end{bmatrix},\ c\neq 0}$$

I selected $\begin{bmatrix}1+i\\-1\end{bmatrix}$ to satisfy the second equation and checked that it also satisfied the first equation. If it had not satisfied the first equation, the system would not have any nonzero solution. This would have been a sure sign of a mechanical error.

The matrix A has real entries. Taking the complex conjugate of $A\vec{v}=\lambda\vec{v}$ then gives $A\bar{\vec{v}}=\bar{\lambda}\bar{\vec{v}}$. If $\vec{v}$ is an eigenvector of eigenvalue λ, then $\bar{\vec{v}}$ is an eigenvector of eigenvalue $\bar{\lambda}$. So the eigenvectors for $\lambda=2-i$ (which is the complex conjugate of $2+i$) should be the complex conjugates of the eigenvectors for $\lambda=2+i$. That is

$$\boxed{c\begin{bmatrix}1-i\\-1\end{bmatrix},\ c\neq 0}$$

As a check, note that

$$\begin{bmatrix}3&2\\-1&1\end{bmatrix}\begin{bmatrix}1+i\\-1\end{bmatrix}=(2+i)\begin{bmatrix}1+i\\-1\end{bmatrix}\qquad\qquad\begin{bmatrix}3&2\\-1&1\end{bmatrix}\begin{bmatrix}1-i\\-1\end{bmatrix}=(2-i)\begin{bmatrix}1-i\\-1\end{bmatrix}$$

2) Find the eigenvalues and eigenvectors of

a) $\begin{bmatrix}2&1&1\\1&2&1\\2&2&3\end{bmatrix}$ b) $\begin{bmatrix}2&1&0\\6&1&-1\\0&0&1\end{bmatrix}$ c) $\begin{bmatrix}2&0&6\\0&3&1\\0&0&3\end{bmatrix}$

Solution. a) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\begin{aligned}\det(A-\lambda I)&=\det\begin{bmatrix}2-\lambda&1&1\\1&2-\lambda&1\\2&2&3-\lambda\end{bmatrix}=\det\begin{bmatrix}2-\lambda&1&1\\-1+\lambda&1-\lambda&0\\-\lambda^2+5\lambda-4&-1+\lambda&0\end{bmatrix}\begin{matrix}(1)\\(2)-(1)\\(3)-(3-\lambda)(1)\end{matrix}\\&=\det\begin{bmatrix}-1+\lambda&1-\lambda\\-\lambda^2+5\lambda-4&-1+\lambda\end{bmatrix}=(\lambda-1)\det\begin{bmatrix}1&-1\\-\lambda^2+5\lambda-4&-1+\lambda\end{bmatrix}\\&=(\lambda-1)[(-1+\lambda)+(-\lambda^2+5\lambda-4)]=(\lambda-1)[-\lambda^2+6\lambda-5]=-(\lambda-1)(\lambda-1)(\lambda-5)\end{aligned}$$

or $\boxed{\lambda=1,1,5}$. The eigenvectors corresponding to $\lambda=1$ are all nonzero solutions of the linear system of equations

$$(A-I)\vec{y}=\begin{bmatrix}1&1&1\\1&1&1\\2&2&2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Longrightarrow\quad\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\boxed{c_1\begin{bmatrix}-1\\0\\1\end{bmatrix}+c_2\begin{bmatrix}0\\-1\\1\end{bmatrix},\ c_1\text{ and }c_2\text{ not both zero}}$$

and those corresponding to $\lambda = 5$ are all nonzero solutions of the linear system of equations

$$(A - 5I)\vec{y} = \begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & -3 & 1 \\ 0 & -8 & 4 \\ 0 & 8 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} (2) \\ (1)+3(2) \\ (3)-2(2) \end{array}$$

$$\implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ c \neq 0}$$

As a check, note that

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

b) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 6 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 6] = (1-\lambda)[\lambda^2 - 3\lambda - 4] = -(\lambda-1)(\lambda-4)(\lambda+1) \end{aligned}$$

or $\boxed{\lambda = -1, 1, 4}$. The eigenvectors corresponding to $\lambda = -1$ are all nonzero solutions of the linear system of equations

$$(A + I)\vec{y} = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \ c \neq 0}$$

those corresponding to $\lambda = 1$ are all nonzero solutions of the linear system of equations

$$(A - I)\vec{y} = \begin{bmatrix} 1 & 1 & 0 \\ 6 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}, \ c \neq 0}$$

and those corresponding to $\lambda = 4$ are all nonzero solutions of the linear system of equations

$$(A - 4I)\vec{y} = \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ c \neq 0}$$

As a check, note that

$$\begin{bmatrix} 2 & 1 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

c) Call the matrix A. The eigenvalues of this matrix are the solutions of

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 0 & 6 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda) \det \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda)(3-\lambda)(2-\lambda) \end{aligned}$$

or $\boxed{\lambda = 2, 3, 3}$. The eigenvectors corresponding to $\lambda = 2$ are all nonzero solutions of the linear system of equations

$$(A - 2I)\vec{y} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ c \neq 0}$$

and those corresponding to $\lambda = 3$ are all nonzero solutions of the linear system of equations

$$(A - 3I)\vec{y} = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \boxed{c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ c \neq 0}$$

Note that there is only a one parameter family of eigenvectors of eigenvalue 3, even though 3 was a double root of $\det(A - \lambda I)$. The usual check:

$$\begin{bmatrix} 2 & 0 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3) Find the functions $x_1(t)$ and $x_2(t)$ satisfying

a) $\begin{aligned} x_1'(t) &= 2x_1(t) + 2x_2(t) \qquad & x_1(0) = 1 \\ x_2'(t) &= 2x_1(t) - \ x_2(t) \qquad & x_2(0) = 0 \end{aligned}$

b) $\begin{aligned} x_1'(t) &= -2x_1(t) + 5x_2(t) \qquad & x_1(0) = \ \ 1 \\ x_2'(t) &= \ \ 4x_1(t) + 6x_2(t) \qquad & x_2(t) = -1 \end{aligned}$

Solution. a) The system of differential equations is of the form $\vec{x}'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions of

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 2 \\ 2 & -1 - \lambda \end{bmatrix} = (2 - \lambda)(-1 - \lambda) - 4 = \lambda^2 - \lambda - 6 = 0$$

or $\lambda = -2, 3$. The eigenvectors corresponding to $\lambda = -2$ are all nonzero solutions of the linear system of equations

$$(A + 2I)\vec{y} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ c \neq 0$$

and those corresponding to $\lambda = 3$ are all nonzero solutions of the linear system of equations

$$(A - 3I)\vec{y} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ c \neq 0$$

Consequently

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}$$

satsifies the differential equations for all values of the constants c_1 and c_2. To satisfy the initial conditions we also need

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2\times 0} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3\times 0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} c_2 &= 2c_1 \\ c_1 + 2c_2 &= 5c_1 = 1 \end{aligned}$$

So $c_1 = \frac{1}{5},\ c_2 = \frac{2}{5}$ and the solution is

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + \tfrac{2}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}}$$

To check, we just sub into the original equations

$$\begin{aligned}
x_1'(t) &= -\tfrac{2}{5}e^{-2t} + \tfrac{12}{5}e^{3t} \\
2x_1(t) + 2x_2(t) &= 2\left\{\tfrac{1}{5}e^{-2t} + \tfrac{4}{5}e^{3t}\right\} + 2\left\{-\tfrac{2}{5}e^{-2t} + \tfrac{2}{5}e^{3t}\right\} = -\tfrac{2}{5}e^{-2t} + \tfrac{12}{5}e^{3t} = x_1'(t) \\
x_1(0) &= \tfrac{1}{5} + \tfrac{4}{5} = 1 \\
x_2'(t) &= \tfrac{4}{5}e^{-2t} + \tfrac{6}{5}e^{3t} \\
2x_1(t) - x_2(t) &= 2\left\{\tfrac{1}{5}e^{-2t} + \tfrac{4}{5}e^{3t}\right\} - \left\{-\tfrac{2}{5}e^{-2t} + \tfrac{2}{5}e^{3t}\right\} = \tfrac{4}{5}e^{-2t} + \tfrac{6}{5}e^{3t} = x_2'(t) \\
x_2(0) &= -\tfrac{2}{5} + \tfrac{2}{5} = 0
\end{aligned}$$

b) The system of differential equations is of the form $\vec{x}'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} -2 & 5 \\ 4 & 6 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions of

$$\det(A - \lambda I) = \det\begin{bmatrix} -2-\lambda & 5 \\ 4 & 6-\lambda \end{bmatrix} = (-2-\lambda)(6-\lambda) - 20 = \lambda^2 - 4\lambda - 32 = 0$$

or $\lambda = -4, 8$. The eigenvectors corresponding to $\lambda = -4$ are all nonzero solutions of the linear system of equations

$$(A + 4I)\vec{y} = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c\begin{bmatrix} 5 \\ -2 \end{bmatrix},\ c \neq 0$$

and those corresponding to $\lambda = 8$ are all nonzero solutions of the linear system of equations

$$(A - 8I)\vec{y} = \begin{bmatrix} -10 & 5 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix},\ c \neq 0$$

Consequently

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{-4t} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{8t}$$

satsifies the differential equations for all values of the constants c_1 and c_2. To satisfy the initial conditions we also need

$$c_1\begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{-4\times 0} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{8\times 0} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Subtracting the second equation from twice the first gives $12c_1 = 3$ or $c_1 = \frac{1}{4}$. Subbing back into the second equation gives $2c_2 = -\frac{1}{2}$ or $c_2 = -\frac{1}{4}$. The solution is

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 5 \\ -2 \end{bmatrix} e^{-4t} - \tfrac{1}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{8t}}$$

To check, we just sub into the original equations

$$\begin{aligned}
x_1'(t) &= -5e^{-4t} - 2e^{8t} \\
-2x_1(t) + 5x_2(t) &= -2\left\{\tfrac{5}{4}e^{-4t} - \tfrac{1}{4}e^{8t}\right\} + 5\left\{-\tfrac{2}{4}e^{-4t} - \tfrac{2}{4}e^{8t}\right\} = -5e^{-4t} - 2e^{8t} = x_1'(t) \\
x_1(0) &= \tfrac{5}{4} - \tfrac{1}{4} = 1 \\
x_2'(t) &= 2e^{-4t} - 4e^{8t} \\
4x_1(t) + 6x_2(t) &= 4\left\{\tfrac{5}{4}e^{-4t} - \tfrac{1}{4}e^{8t}\right\} + 6\left\{-\tfrac{2}{4}e^{-4t} - \tfrac{2}{4}e^{8t}\right\} = 2e^{-4t} - 4e^{8t} = x_2'(t) \\
x_2(0) &= -\tfrac{2}{4} - \tfrac{2}{4} = -1
\end{aligned}$$

4) Find the functions $x_1(t)$, $x_2(t)$ and $x_3(t)$ satisfying

$$\begin{aligned} x_1'(t) &= 2x_1(t) + x_2(t) & \qquad x_1(0) &= 1 \\ x_2'(t) &= 6x_1(t) + x_2(t) - x_3(t) & \qquad x_2(0) &= 2 \\ x_3'(t) &= \phantom{6x_1(t) + x_2(t) -{}} x_3(t) & \qquad x_3(0) &= 3 \end{aligned}$$

Solution. The system of differential equations is of the form $\vec{x}\,'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 6 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions of

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 6 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Expanding along the third row

$$\begin{aligned} 0 = \det(A - \lambda I) &= (1-\lambda)\det \begin{bmatrix} 2-\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)\big[(2-\lambda)(1-\lambda) - 6\big] = (1-\lambda)\big[\lambda^2 - 3\lambda - 4\big] \\ &= (1-\lambda)(\lambda-4)(\lambda+1) \end{aligned}$$

or $\lambda = -1, 1, 4$. The eigenvectors corresponding to, in order, $\lambda = -1, 1, 4$ are all nonzero solutions of the linear systems of equations

$$\begin{aligned} (A+I)\vec{y} &= \begin{bmatrix} 3 & 1 & 0 \\ 6 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix},\ c_1 \neq 0 \\ (A-I)\vec{y} &= \begin{bmatrix} 1 & 1 & 0 \\ 6 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix},\ c_2 \neq 0 \\ (A-4I)\vec{y} &= \begin{bmatrix} -2 & 1 & 0 \\ 6 & -3 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ c_3 \neq 0 \end{aligned}$$

Consequently

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{4t}$$

satsifies the differential equations for all values of the constants c_1, c_2 and c_3. To satisfy the initial conditions we also need

$$c_1 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & 2 \\ 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The last equation forces $c_2 = \frac{1}{2}$. Subbing this into the first two equations gives

$$\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 5/2 \end{bmatrix}$$

Adding three times equation (1) to equation (2) gives $5c_3 = 4$ or $c_3 = \frac{4}{5}$ and then equation (1) gives $c_1 = -\frac{3}{10}$. The solution is

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = -\tfrac{3}{10}\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} e^{-t} + \tfrac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} e^{t} + \tfrac{4}{5}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{4t}}$$

To check, we just sub into the original equations

$$\begin{aligned}
x_1'(t) &= \tfrac{3}{10}e^{-t} + \tfrac{1}{2}e^{t} + \tfrac{16}{5}e^{4t} \\
2x_1(t) + x_2(t) &= 2\left\{-\tfrac{3}{10}e^{-t} + \tfrac{1}{2}e^{t} + \tfrac{4}{5}e^{4t}\right\} + \left\{\tfrac{9}{10}e^{-t} - \tfrac{1}{2}e^{t} + \tfrac{8}{5}e^{4t}\right\} = \tfrac{3}{10}e^{-t} + \tfrac{1}{2}e^{t} + \tfrac{16}{5}e^{4t} = x_1'(t) \\
x_1(0) &= -\tfrac{3}{10} + \tfrac{1}{2} + \tfrac{4}{5} = 1 \\
x_2'(t) &= -\tfrac{9}{10}e^{-t} - \tfrac{1}{2}e^{t} + \tfrac{32}{5}e^{4t} \\
6x_1(t) + x_2(t) - x_3(t) &= 6\left\{-\tfrac{3}{10}e^{-t} + \tfrac{1}{2}e^{t} + \tfrac{4}{5}e^{4t}\right\} + \left\{\tfrac{9}{10}e^{-t} - \tfrac{1}{2}e^{t} + \tfrac{8}{5}e^{4t}\right\} - 3e^{t} \\
&= -\tfrac{9}{10}e^{-t} - \tfrac{1}{2}e^{t} + \tfrac{32}{5}e^{4t} = x_2'(t) \\
x_2(0) &= \tfrac{9}{10} - \tfrac{1}{2} + \tfrac{8}{5} = 2 \\
x_3'(t) &= 3e^{t} \\
x_3(t) &= 3e^{t} = x_3'(t) \\
x_3(0) &= 3
\end{aligned}$$

5) Find the functions $x_1(t)$, $x_2(t)$ and $x_3(t)$ satisfying

$$\begin{aligned}
x_1'(t) &= 2x_1(t) - 6x_2(t) - 6x_3(t) &\qquad x_1(0) &= 0 \\
x_2'(t) &= -x_1(t) + x_2(t) + 2x_3(t) &\qquad x_2(0) &= 1 \\
x_3'(t) &= 3x_1(t) - 6x_2(t) - 7x_3(t) &\qquad x_3(0) &= 0
\end{aligned}$$

Solution. The system of differential equations is of the form $\vec{x}'(t) = A\vec{x}(t)$ with

$$A = \begin{bmatrix} 2 & -6 & -6 \\ -1 & 1 & 2 \\ 3 & -6 & -7 \end{bmatrix}$$

The eigenvalues of this matrix are the solutions of

$$\begin{aligned}
0 = \det(A - \lambda I) &= \det\begin{bmatrix} 2-\lambda & -6 & -6 \\ -1 & 1-\lambda & 2 \\ 3 & -6 & -7-\lambda \end{bmatrix} \\
&= \det\begin{bmatrix} 0 & -4-3\lambda+\lambda^2 & -2-2\lambda \\ -1 & 1-\lambda & 2 \\ 0 & -3-3\lambda & -1-\lambda \end{bmatrix} \begin{matrix} (1)+(2-\lambda)(2) \\ (2) \\ (3)+3(2) \end{matrix} \\
&= \det\begin{bmatrix} -4-3\lambda+\lambda^2 & -2-2\lambda \\ -3-3\lambda & -1-\lambda \end{bmatrix} \\
&= (-4-3\lambda+\lambda^2)(-1-\lambda) - (-2-2\lambda)(-3-3\lambda) \\
&= (-4+\lambda)(1+\lambda)(-1-\lambda) - 6(1+\lambda)(1+\lambda) \\
&= (1+\lambda)^2[4-\lambda-6] = (1+\lambda)^2[-2-\lambda]
\end{aligned}$$

or $\lambda=-1,-1,-2$. The eigenvectors corresponding to, in order, $\lambda=-1,-2$ are all nonzero solutions of the linear systems of equations

$$(A+I)\vec{y}=\begin{bmatrix}3&-6&-6\\-1&2&2\\3&-6&-6\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Longrightarrow\quad\begin{bmatrix}-1&2&2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{array}{c}(2)\\(1)+3(2)\\(3)+3(2)\end{array}$$

$$\Longrightarrow\quad\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=c_1\begin{bmatrix}2\\1\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\1\end{bmatrix},\ c_1,c_2\text{ not both zero}$$

$$(A+2I)\vec{y}=\begin{bmatrix}4&-6&-6\\-1&3&2\\3&-6&-5\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Longrightarrow\quad\begin{bmatrix}0&6&2\\-1&3&2\\0&3&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\begin{array}{c}(1)+4(2)\\(2)\\(3)+3(2)\end{array}$$

$$\Longrightarrow\quad\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=c_3\begin{bmatrix}3\\-1\\3\end{bmatrix},\ c_3\neq 0$$

Consequently

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=c_1\begin{bmatrix}2\\1\\0\end{bmatrix}e^{-t}+c_2\begin{bmatrix}2\\0\\1\end{bmatrix}e^{-t}+c_3\begin{bmatrix}3\\-1\\3\end{bmatrix}e^{-2t}$$

satsifies the differential equations for all values of the constants c_1, c_2 and c_3. To satisfy the initial conditions we also need

$$c_1\begin{bmatrix}2\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}3\\-1\\3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}\ \text{or}\ \begin{bmatrix}2&2&3\\1&0&-1\\0&1&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}\ \text{or}\ \begin{bmatrix}0&0&-1\\1&0&-1\\0&1&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

In the last step we replaced (1) by (1) $-$ 2(2) $-$ 2(3). The solution is $c_3=2,\ c_1=3,\ c_2=-6$ so

$$\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=3\begin{bmatrix}2\\1\\0\end{bmatrix}e^{-t}-6\begin{bmatrix}2\\0\\1\end{bmatrix}e^{-t}+2\begin{bmatrix}3\\-1\\3\end{bmatrix}e^{-2t}=\boxed{\begin{bmatrix}-6\\3\\-6\end{bmatrix}e^{-t}+\begin{bmatrix}6\\-2\\6\end{bmatrix}e^{-2t}}$$

To check, we just sub into the original equations

$$\begin{aligned}x_1'(t)&=6e^{-t}-12e^{-2t}\\2x_1(t)-6x_2(t)-6x_3(t)&=2\left\{-6e^{-t}+6e^{-2t}\right\}-6\left\{3e^{-t}-2e^{-2t}\right\}-6\left\{-6e^{-t}+6e^{-2t}\right\}=x_1'(t)\\x_1(0)&=-6+6=0\end{aligned}$$

$$\begin{aligned}x_2'(t)&=-3e^{-t}+4e^{-2t}\\-x_1(t)+x_2(t)+2x_3(t)&=-\left\{-6e^{-t}+6e^{-2t}\right\}+\left\{3e^{-t}-2e^{-2t}\right\}+2\left\{-6e^{-t}+6e^{-2t}\right\}=x_2'(t)\\x_2(0)&=3-2=1\end{aligned}$$

$$\begin{aligned}x_3'(t)&=6e^{-t}-12e^{-2t}\\3x_1(t)-6x_2(t)-7x_3(t)&=3\left\{-6e^{-t}+6e^{-2t}\right\}-6\left\{3e^{-t}-2e^{-2t}\right\}-7\left\{-6e^{-t}+6e^{-2t}\right\}=x_3'(t)\\x_3(0)&=-6+6=0\end{aligned}$$

6) Let

$$A=\begin{bmatrix}-3&0&2\\1&-1&0\\-2&-1&0\end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution.

$$\det(A-\lambda I)=\det\begin{bmatrix}-3-\lambda & 0 & 2\\ 1 & -1-\lambda & 0\\ -2 & -1 & -\lambda\end{bmatrix}=(-3-\lambda)\det\begin{bmatrix}-1-\lambda & 0\\ -1 & -\lambda\end{bmatrix}+2\det\begin{bmatrix}1 & -1-\lambda\\ -2 & -1\end{bmatrix}$$
$$=(-3-\lambda)(-1-\lambda)(-\lambda)+2[-1-2-2\lambda]=-\lambda^3-4\lambda^2-3\lambda-6-4\lambda$$
$$=-\lambda^3-4\lambda^2-7\lambda-6$$

By Trick # 1 of Appendix A, any integer root of $-\lambda^3-4\lambda^2-7\lambda-6$ must divide the constant term -6. So, the only candidates for integer roots of this polynomial are $\pm1,\ \pm2,\ \pm3,\ \pm6$. Subbing $\lambda=1$ into $-\lambda^3-4\lambda^2-7\lambda-6$ gives a sum of four strictly negative terms. So $\lambda=1$ cannot be a root. The same argument rules out $\lambda=2,\ 3$. We try the three negative candidates

$$-\lambda^3-4\lambda^2-7\lambda-6\big|_{\lambda=-1}=-2$$
$$-\lambda^3-4\lambda^2-7\lambda-6\big|_{\lambda=-2}=0$$
$$-\lambda^3-4\lambda^2-7\lambda-6\big|_{\lambda=-3}=6$$
$$-\lambda^3-4\lambda^2-7\lambda-6\big|_{\lambda=-6}=108$$

We have found one root so far, namely $\lambda=-2$. To find the other two, factor $(\lambda+2)$ out of the determinant (see Trick # 3 of Appendix A):

$$\det(A-\lambda I)=-\lambda^3-4\lambda^2-7\lambda-6=(\lambda+2)(-\lambda^2-2\lambda-3)$$

The roots of $\lambda^2+2\lambda+3$ are $\frac{1}{2}(-2\pm\sqrt{4-12})$ so the eigenvalues are $-2,-1+\sqrt{2}i,-1-\sqrt{2}i$.

$$\text{For }\lambda=-2,\qquad A-\lambda I=\begin{bmatrix}-1 & 0 & 2\\ 1 & 1 & 0\\ -2 & -1 & 2\end{bmatrix}\qquad\Longrightarrow\qquad\text{eigenvectors } c\begin{bmatrix}2\\ -2\\ 1\end{bmatrix}$$
$$\text{For }\lambda=-1+\sqrt{2}i,\ A-\lambda I=\begin{bmatrix}-2-\sqrt{2}i & 0 & 2\\ 1 & -\sqrt{2}i & 0\\ -2 & -1 & 1-\sqrt{2}i\end{bmatrix}\quad\Longrightarrow\quad\text{eigenvectors } c\begin{bmatrix}\sqrt{2}i\\ 1\\ -1+\sqrt{2}i\end{bmatrix}$$

with c taking any nonzero value as usual. The final eigenvalue and eigenvector may be obtained by taking complex conjugates of the $\lambda=-1+\sqrt{2}i$ eigenvalue and eigenvector.

$$\boxed{\lambda_1=-2\quad \vec v_1=\begin{bmatrix}2\\ -2\\ 1\end{bmatrix}\qquad \lambda_2=-1+\sqrt{2}i\quad \vec v_2=\begin{bmatrix}\sqrt{2}i\\ 1\\ -1+\sqrt{2}i\end{bmatrix}\qquad \lambda_3=-1-\sqrt{2}i\quad \vec v_3=\begin{bmatrix}-\sqrt{2}i\\ 1\\ -1-\sqrt{2}i\end{bmatrix}}$$

7) Consider the differential equation $\vec x'(t)=A\vec x(t)$ with

$$A=\begin{bmatrix}-3 & 0 & 2\\ 1 & -1 & 0\\ -2 & -1 & 0\end{bmatrix}$$

Find (a) the general solution and (b) the solution that satisfies the initial conditions $x_1(0)=2,\ x_2(0)=0,\ x_3(0)=-1$.

Solution 1. (Using complex exponentials) The general solution is, using the eigenvalues and eigenvectors computed in Problem 6

$$\boxed{\vec x(t)=\alpha e^{-2t}\begin{bmatrix}2\\ -2\\ 1\end{bmatrix}+\beta e^{(-1+\sqrt{2}i)t}\begin{bmatrix}\sqrt{2}i\\ 1\\ -1+\sqrt{2}i\end{bmatrix}+\gamma e^{(-1-\sqrt{2}i)t}\begin{bmatrix}-\sqrt{2}i\\ 1\\ -1-\sqrt{2}i\end{bmatrix}}$$

To satisfy the initial conditions, we need

$$\vec{x}(0) = \alpha \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} \sqrt{2}i \\ 1 \\ -1+\sqrt{2}i \end{bmatrix} + \gamma \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

We solve for $\alpha,\ \beta,\ \gamma$ by row reduction as usual

$$\left[\begin{array}{ccc|c} 2 & \sqrt{2}i & -\sqrt{2}i & 2 \\ -2 & 1 & 1 & 0 \\ 1 & -1+\sqrt{2}i & -1-\sqrt{2}i & -1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & \sqrt{2}i & -\sqrt{2}i & 2 \\ 0 & 1+\sqrt{2}i & 1-\sqrt{2}i & 2 \\ 0 & -2+\sqrt{2}i & -2-\sqrt{2}i & -4 \end{array}\right] \begin{array}{c} (1) \\ (2)+(1) \\ 2(3)-(1) \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & \sqrt{2}i & -\sqrt{2}i & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -2+\sqrt{2}i & -2-\sqrt{2}i & -4 \end{array}\right] \begin{array}{c} (1) \\ [(2)-(3)]/3 \\ (3) \end{array} \left[\begin{array}{ccc|c} 2 & \sqrt{2}i & -\sqrt{2}i & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2\sqrt{2}i & -2\sqrt{2}i \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3)+(2-\sqrt{2}i)(2) \end{array}$$

$$\implies \quad \gamma = 1,\ \beta = 1,\ \alpha = 1$$

So

$$\boxed{\vec{x}(t) = e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + e^{(-1+\sqrt{2}i)t} \begin{bmatrix} \sqrt{2}i \\ 1 \\ -1+\sqrt{2}i \end{bmatrix} + e^{(-1-\sqrt{2}i)t} \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i \end{bmatrix}}$$

We can, if we wish, convert this into trig functions by subbing in

$$e^{\sqrt{2}\,ti} = \cos(\sqrt{2}\,t) + i\sin(\sqrt{2}\,t) \qquad e^{-\sqrt{2}\,ti} = \cos(\sqrt{2}\,t) - i\sin(\sqrt{2}\,t)$$

To do so, we first factor $e^{(-1\pm\sqrt{2}i)t} = e^{-t}e^{\pm\sqrt{2}\,ti}$.

$$\begin{aligned} \vec{x}(t) &= e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} \sqrt{2}ie^{\sqrt{2}\,ti} \\ e^{\sqrt{2}\,ti} \\ -e^{\sqrt{2}\,ti} + \sqrt{2}ie^{\sqrt{2}\,ti} \end{bmatrix} + e^{-t} \begin{bmatrix} -\sqrt{2}ie^{-\sqrt{2}\,ti} \\ e^{-\sqrt{2}\,ti} \\ -e^{-\sqrt{2}\,ti} - \sqrt{2}ie^{-\sqrt{2}\,ti} \end{bmatrix} \\ &= e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} \sqrt{2}ie^{\sqrt{2}\,ti} - \sqrt{2}ie^{-\sqrt{2}\,ti} \\ e^{\sqrt{2}\,ti} + e^{-\sqrt{2}\,ti} \\ -e^{\sqrt{2}\,ti} + \sqrt{2}ie^{\sqrt{2}\,ti} - e^{-\sqrt{2}\,ti} - \sqrt{2}ie^{-\sqrt{2}\,ti} \end{bmatrix} \\ &= \boxed{e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} -2\sqrt{2}\sin(\sqrt{2}\,t) \\ 2\cos(\sqrt{2}\,t) \\ -2\cos(\sqrt{2}\,t) - 2\sqrt{2}\sin(\sqrt{2}\,t) \end{bmatrix}} \end{aligned}$$

Solution 2. (converting to sin's and cos's) The general solution is, using the eigenvalues and eigenvectors computed in Problem 6,

$$\vec{x}(t) = \alpha e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \beta e^{(-1+\sqrt{2}i)t} \begin{bmatrix} \sqrt{2}i \\ 1 \\ -1+\sqrt{2}i \end{bmatrix} + \gamma e^{(-1-\sqrt{2}i)t} \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1-\sqrt{2}i \end{bmatrix}$$

This time, we convert the general solution directly into trig functions. Define two new arbitrary constants b and c by

$$\beta = \tfrac{1}{2}(b + \tfrac{1}{i}c)$$
$$\gamma = \tfrac{1}{2}(b - \tfrac{1}{i}c)$$

The definitions are rigged so that the net coefficients of b and c in $\vec{x}(t)$ are precisely the real and imaginary parts of

$$e^{(-1+\sqrt{2}i)t}\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix}$$

respectively, as we see in the second and third lines of

$$\begin{aligned}
\vec{x}(t) &= \alpha e^{-2t}\begin{bmatrix}2\\-2\\1\end{bmatrix} + \tfrac{1}{2}(b+\tfrac{1}{i}c)e^{(-1+\sqrt{2}i)t}\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix} + \tfrac{1}{2}(b-\tfrac{1}{i}c)e^{(-1-\sqrt{2}i)t}\begin{bmatrix}-\sqrt{2}i\\1\\-1-\sqrt{2}i\end{bmatrix}\\
&= \alpha e^{-2t}\begin{bmatrix}2\\-2\\1\end{bmatrix} + b\frac{1}{2}\left(e^{(-1+\sqrt{2}i)t}\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix} + e^{(-1-\sqrt{2}i)t}\begin{bmatrix}-\sqrt{2}i\\1\\-1-\sqrt{2}i\end{bmatrix}\right)\\
&\qquad + c\frac{1}{2i}\left(e^{(-1+\sqrt{2}i)t}\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix} - e^{(-1-\sqrt{2}i)t}\begin{bmatrix}-\sqrt{2}i\\1\\-1-\sqrt{2}i\end{bmatrix}\right)\\
&= \alpha e^{-2t}\begin{bmatrix}2\\-2\\1\end{bmatrix} + be^{-t}\frac{1}{2}\left([\cos(\sqrt{2}\,t)+i\sin(\sqrt{2}\,t)]\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix} + [\cos(\sqrt{2}\,t)-i\sin(\sqrt{2}\,t)]\begin{bmatrix}-\sqrt{2}i\\1\\-1-\sqrt{2}i\end{bmatrix}\right)\\
&\qquad + ce^{-t}\frac{1}{2i}\left([\cos(\sqrt{2}\,t)+i\sin(\sqrt{2}\,t)]\begin{bmatrix}\sqrt{2}i\\1\\-1+\sqrt{2}i\end{bmatrix} - [\cos(\sqrt{2}\,t)-i\sin(\sqrt{2}\,t)]\begin{bmatrix}-\sqrt{2}i\\1\\-1-\sqrt{2}i\end{bmatrix}\right)\\
&= \boxed{\alpha e^{-2t}\begin{bmatrix}2\\-2\\1\end{bmatrix} + be^{-t}\begin{bmatrix}-\sqrt{2}\sin(\sqrt{2}\,t)\\\cos(\sqrt{2}\,t)\\-\cos(\sqrt{2}\,t)-\sqrt{2}\sin(\sqrt{2}\,t)\end{bmatrix} + ce^{-t}\begin{bmatrix}\sqrt{2}\cos(\sqrt{2}\,t)\\\sin(\sqrt{2}\,t)\\-\sin(\sqrt{2}\,t)+\sqrt{2}\cos(\sqrt{2}\,t)\end{bmatrix}}
\end{aligned}$$

To satisfy the initial conditions, we need

$$\vec{x}(0) = \alpha\begin{bmatrix}2\\-2\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix} + c\begin{bmatrix}\sqrt{2}\\0\\\sqrt{2}\end{bmatrix} = \begin{bmatrix}2\\0\\-1\end{bmatrix}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c}2&0&\sqrt{2}&2\\-2&1&0&0\\1&-1&\sqrt{2}&-1\end{array}\right]\qquad \left[\begin{array}{ccc|c}2&0&\sqrt{2}&2\\0&1&\sqrt{2}&2\\0&-1&2\sqrt{2}&-2\end{array}\right]\begin{array}{c}(1)\\(2)+(1)\\2(3)-(1)\end{array}\qquad \left[\begin{array}{ccc|c}2&0&\sqrt{2}&2\\0&1&\sqrt{2}&2\\0&0&3\sqrt{2}&0\end{array}\right]\begin{array}{c}(1)\\(2)\\(2)+(3)\end{array}$$

so that $c=0,\ b=2,\ \alpha=1$ and

$$\boxed{\vec{x}(t) = e^{-2t}\begin{bmatrix}2\\-2\\1\end{bmatrix} + 2e^{-t}\begin{bmatrix}-\sqrt{2}\sin(\sqrt{2}\,t)\\\cos(\sqrt{2}\,t)\\-\cos(\sqrt{2}\,t)-\sqrt{2}\sin(\sqrt{2}\,t)\end{bmatrix}}$$

8) State whether each of the following statments is true or false. In each case give a *brief* reason.

a) The matrix

$$\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

has no eigenvectors.

b) The vector $\begin{bmatrix}0\\0\end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

c) If λ is an eigenvalue of the matrix A, then λ^3 is a eigenvalue of the matrix A^3.

d) If 0 is an eigenvalue of the matrix A, then A is not invertible.

Solution. a) This statement is $\boxed{\text{false}}$. *Every* square matrix has at least one eigenvector. For the given matrix, the vector $[1,0,0]^t$ is an eigenvector of eigenvalue 0.

b) This statement is $\boxed{\text{false}}$. By definition, the zero vector is *never* an eigenvector.

c) This statement is $\boxed{\text{true}}$. Let $\vec{v}$ be an eigenvector of A of eigenvalue λ. Then

$$A^3\vec{v} = A^2(A\vec{v}) = A^2(\lambda\vec{v}) = \lambda A^2\vec{v} = \lambda A(A\vec{v}) = \lambda A(\lambda\vec{v}) = \lambda^2 A\vec{v} = \lambda^3\vec{v}$$

which shows that $\vec{v}$ is an eigenvector of A^3 of eigenvalue λ^3.

d) This statement is $\boxed{\text{true}}$. Let $\vec{v}$ be an eigenvector of A of eigenvalue 0. Then $A\vec{v} = 0\vec{v} = \vec{0}$. If A^{-1} were to exist, then $\vec{v} = A^{-1}A\vec{v} = A^{-1}\vec{0} = \vec{0}$. But $\vec{0}$ may never be an eigenvector.

9) Find, if possible, a matrix A obeying

$$A^3 = \begin{bmatrix}-34 & -105\\ 14 & 43\end{bmatrix}$$

Solution. Call the given matrix B. We shall implement the following strategy. First, we shall diagonalize B. That is, find matrices U and D, with U invertible and D diagonal so that $B = UDU^{-1}$. Then we shall find a matrix F obeying $F^3 = D$. This will be easy, because D is diagonal. Finally we shall define $A = UFU^{-1}$ and observe that, as desired,

$$A^3 = \left(UFU^{-1}\right)^3 = UFU^{-1}UFU^{-1}UFU^{-1} = UFIFIFU^{-1} = UF^3U^{-1} = UDU^{-1} = B$$

To diagonalize B, we find the eigenvalues and eigenvectors of B.

$$\det\begin{bmatrix}-34-\lambda & -105\\ 14 & 43-\lambda\end{bmatrix} = \lambda^2 - (43-34)\lambda + (-34\times 43 + 105\times 14) = \lambda^2 - 9\lambda + 8 = (\lambda-1)(\lambda-8)$$

The eigenvectors of eigenvalue 1 are the nonzero solutions of

$$\begin{bmatrix}-35 & -105\\ 14 & 42\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Any nonzero multiple of $\begin{bmatrix}3\\-1\end{bmatrix}$ is an eigenvector of eigenvalue 1. The eigenvectors of eigenvalue 8 are the nonzero solutions of

$$\begin{bmatrix}-42 & -105\\ 14 & 35\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Any nonzero multiple of $\begin{bmatrix}5\\-2\end{bmatrix}$ is an eigenvector of eigenvalue 8. Denote by

$$D = \begin{bmatrix}1 & 0\\ 0 & 8\end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix}3 & 5\\ -1 & -2\end{bmatrix}$$

the eigenvalue and eignevector matrices, respectively for B. Then we should have $B = UDU^{-1}$. As a check that our eigenvalues and eigenvectors are correct, we compute, using the canned formula

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

for the inverse of a 2×2 matrix derived in Example III.17, that

$$U^{-1} = \tfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix} = -\begin{bmatrix} -2 & -5 \\ 1 & 3\end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & -3\end{bmatrix}$$

and hence

$$\begin{aligned} UDU^{-1} &= \begin{bmatrix} 3 & 5 \\ -1 & -2\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 8\end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3\end{bmatrix} \\ &= \begin{bmatrix} 3 & 5 \\ -1 & -2\end{bmatrix}\begin{bmatrix} 2 & 5 \\ -8 & -24\end{bmatrix} \\ &= \begin{bmatrix} -34 & -105 \\ 14 & 43\end{bmatrix} = B \end{aligned}$$

as desired. We finally determine the cube root, by observing that

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix}$$

obeys $F^3 = D$. Set

$$A = UFU^{-1} = \begin{bmatrix} 3 & 5 \\ -1 & -2\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2\end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -3\end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & -2\end{bmatrix}\begin{bmatrix} 2 & 5 \\ -2 & -6\end{bmatrix} = \boxed{\begin{bmatrix} -4 & -15 \\ 2 & 7\end{bmatrix}}$$

To check that this is correct, we multiply out

$$\begin{bmatrix} -4 & -15 \\ 2 & 7\end{bmatrix}\begin{bmatrix} -4 & -15 \\ 2 & 7\end{bmatrix}\begin{bmatrix} -4 & -15 \\ 2 & 7\end{bmatrix} = \begin{bmatrix} -14 & -45 \\ 6 & 19\end{bmatrix}\begin{bmatrix} -4 & -15 \\ 2 & 7\end{bmatrix} = \begin{bmatrix} -34 & -105 \\ 14 & 43\end{bmatrix}$$

10) Find a 3×3 matrix M having the eigenvalues 1 and 2, such that the eigenspace for $\lambda = 1$ is a line whose direction vector is $[2,0,1]$ and the eigenspace for $\lambda = 2$ is the plane $x - 2y + z = 0$.

Solution. The plane $x - 2y + z = 0$ contains the vectors $[1,1,1]$ and $[0,1,2]$. We want $[1,1,1]$ and $[0,1,2]$ to be eigenvectors of eigenvalue 2 and $[2,0,1]$ to be an eigenvector of eigenvalue 1. Define

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1\end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1\end{bmatrix}$$

Then $M = UDU^{-1}$. To evaluate it, we first find the inverse of U.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1\end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-(1) \\ (3)-(1)\end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1\end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-2(2)\end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 & -2 & 1\end{array}\right] \qquad \begin{array}{r} (1)-2(3)/3 \\ (2)+2(3)/3 \\ (3)/3\end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 4/3 & -2/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & -2/3 & 1/3\end{array}\right]$$

Now

$$\begin{aligned} M = UDU^{-1} &= \frac{1}{3}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 4 & -2 \\ -1 & -1 & 2 \\ 1 & -2 & 1\end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix} 2 & 8 & -4 \\ -2 & -2 & 4 \\ 1 & -2 & 1\end{bmatrix} \\ &= \boxed{\frac{1}{3}\begin{bmatrix} 4 & 4 & -2 \\ 0 & 6 & 0 \\ -1 & 2 & 5\end{bmatrix}} \end{aligned}$$

As a check, observe that

$$\frac{1}{3}\begin{bmatrix} 4 & 4 & -2 \\ 0 & 6 & 0 \\ -1 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

and, for any x and y, setting $z = -x + 2y$ so that (x, y, z) lies on the plane $x - 2y + z = 0$,

$$\frac{1}{3}\begin{bmatrix} 4 & 4 & -2 \\ 0 & 6 & 0 \\ -1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ -x+2y \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ -x+2y \end{bmatrix}$$

11) Consider a population which is divided into three types and reproduces as follows:
70% of the offspring of type 1 are type 1, 10% are type 2 and 20% are type 3
10% of the offspring of type 2 are type 1, 80% are type 2 and 10% are type 3
10% of the offspring of type 3 are type 1, 30% are type 2 and 60% are type 3
All three types reproduce at the same rate. Let $x_i(n)$ denote the fraction of generation n which is of type i, for $i = 1, 2, 3$.
a) Find a matrix A such that $\vec{x}(n+1) = A\vec{x}(n)$.
b) Find the eigenvalues of A.
c) Is there an equilibrium distribution, i.e. a vector $\vec{x}$ such that $\vec{x}(n) = \vec{x}$ for all n if $\vec{x}(0) = \vec{x}$? If so, find it.

Solution. a) From the reproduction rules

$$\begin{aligned} x_1(n+1) &= .7x_1(n) + .1x_2(n) + .1x_3(n) \\ x_2(n+1) &= .1x_1(n) + .8x_2(n) + .3x_3(n) \\ x_3(n+1) &= .2x_1(n) + .1x_2(n) + .6x_3(n) \end{aligned}$$

The desired matrix is

$$\boxed{A = \begin{bmatrix} .7 & .1 & .1 \\ .1 & .8 & .3 \\ .2 & .1 & .6 \end{bmatrix}}$$

b) The eigenvalues are the roots of

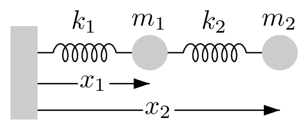
$$\begin{aligned} \det(A - \lambda I) &= \det\begin{bmatrix} .7-\lambda & .1 & .1 \\ .1 & .8-\lambda & .3 \\ .2 & .1 & .6-\lambda \end{bmatrix} \\ &= (.7-\lambda)\det\begin{bmatrix} .8-\lambda & .3 \\ .1 & .6-\lambda \end{bmatrix} - .1\det\begin{bmatrix} .1 & .3 \\ .2 & .6-\lambda \end{bmatrix} + .1\det\begin{bmatrix} .1 & .8-\lambda \\ .2 & .1 \end{bmatrix} \\ &= (.7-\lambda)[(.8-\lambda)(.6-\lambda) - .03] - .1[.1(.6-\lambda) - .06] + .1[.01 - .2(.8-\lambda)] \\ &= (.7-\lambda)[\lambda^2 - 1.4\lambda + .45] + .01\lambda + .02\lambda - .015 \\ &= -\lambda^3 + 2.1\lambda^2 - 1.4\lambda + .3 \\ &= -(\lambda-1)(\lambda^2 - 1.1\lambda + .3) \\ &= -(\lambda-1)(\lambda-.5)(\lambda-.6) \end{aligned}$$

The eigenvalues are $\boxed{1, .5, .6}$.
c) There is a eigenvector of eigenvalue 1. This vector will not change when multplied by A. The eigenvector of eigenvalue 1 is a nonzero solution of

$$\begin{bmatrix} -.3 & .1 & .1 \\ .1 & -.2 & .3 \\ .2 & .1 & -.4 \end{bmatrix}\vec{v} = \vec{0} \quad\Longrightarrow\quad \begin{array}{c} (1)+3(2) \\ (2) \\ (3)-2(2) \end{array}\begin{bmatrix} 0 & -.5 & 1 \\ .1 & -.2 & .3 \\ 0 & .5 & -1 \end{bmatrix}\vec{v} = \vec{0} \quad\Longrightarrow\quad \vec{v} = \alpha\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The elements of the equilibrium distribution must add up to 1, so we choose $\alpha = 1/4$ giving $\boxed{\begin{bmatrix} .25 \\ .5 \\ .25 \end{bmatrix}}$.

12) Consider the following mass-spring system on a frictionless plane. Both masses are 1 kg. and the natural



length of both springs is 1 m. Their spring constants are k_1 and k_2. Let $x_i,\ i = 1, 2$ denote the distance of mass i from the wall at the left.

a) Use Newton's law to find the equations of motion of the masses.

b) Write these equations as a first order system of differential equations.

Solution. a) The forces exerted by the two springs are $k_1(x_1 - \ell_1)$ and $k_2(x_2 - x_1 - \ell_2)$, with $\ell_1 = \ell_2 = 1$, so Newton says

$$\begin{aligned} x_1'' &= k_2(x_2 - x_1 - 1) - k_1(x_1 - 1) \\ x_2'' &= -k_2(x_2 - x_1 - 1) \end{aligned}$$

since $m_1 = m_2 = 1$.

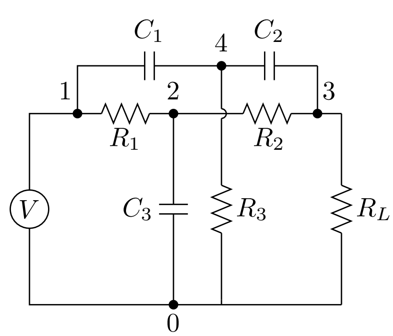
b) Define $x_3 = x_1'$ and $x_4 = x_2'$. Then, since $x_3' = x_1''$ and $x_4' = x_2''$,

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_4 \\ x_3' &= k_2(x_2 - x_1 - 1) - k_1(x_1 - 1) \\ x_4' &= -k_2(x_2 - x_1 - 1) \end{aligned}$$

This is a first order system of differential equations. But it is not homogeneous because of the constant terms, $-k_2 + k_1$ and k_2 that appear in equations three and four. If you prefer to have a homogeneous first order system (this is not required by the statement of the problem), just change variables from x_1 and x_2 to $y_1 = x_1 - 1$ and $y_2 = x_2 - 2$. These coordinates are the positions of the masses measured relative to their equilibrium positions. (In equilibrium, $x_1 = \ell_1$ and $x_2 = \ell_2$.) Then,

$$\begin{aligned} y_1' &= x_3 \\ y_2' &= x_4 \\ x_3' &= k_2(y_2 - y_1) - k_1 y_1 \\ x_4' &= -k_2(y_2 - y_1) \end{aligned}$$

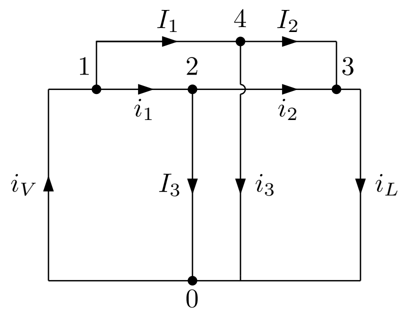
13) The circuit below is known as a twin–T RC network and is used as a filter. Find a system of equations that determine the various currents flowing in the circuit, asssuming that the applied voltage, $V = V(t)$, is given.



Solution. Denote by Q_1, Q_2, Q_3 and I_1, I_2, I_3 the charges on and currents flowing (rightward or downward) through the capacitors C_1, C_2, C_3, respectively. These charges and currents are related by

$$I_1 = \tfrac{dQ_1}{dt} \qquad I_2 = \tfrac{dQ_2}{dt} \qquad I_3 = \tfrac{dQ_3}{dt}$$

Denote by i_1, i_2, i_3, i_L the currents flowing (rightward or downward) through R_1, R_2, R_3, R_L respectively and by i_V the current flowing (upward) through V. Here is a figure showing the currents flowing in the circuit.



By Kirchhoff's current law applied at the nodes 0 (actually 0 and the node immediately to its right), 1, 2, 3 and 4 gives

$$\begin{aligned}
&(0) \qquad i_V = I_3 + i_3 + i_L \\
&(1) \qquad i_V = I_1 + i_1 \\
&(2) \qquad i_1 = I_3 + i_2 \\
&(3) \qquad i_L = I_2 + i_2 \\
&(4) \qquad I_1 = I_2 + i_3
\end{aligned}$$

We can easily use these equations to express all currents in terms of, for example, i_1, i_2, i_3, i_L:

$$\begin{aligned}
(3) &\quad\Longrightarrow\quad I_2 = i_L - i_2 \\
(2) &\quad\Longrightarrow\quad I_3 = i_1 - i_2 \\
(4) &\quad\Longrightarrow\quad I_1 = I_2 + i_3 = i_L - i_2 + i_3 \\
(1) &\quad\Longrightarrow\quad i_V = I_1 + i_1 = i_L + i_1 - i_2 + i_3
\end{aligned}$$

The one equation that we did not use here, $i_V = I_3 + i_3 + i_L$, is redundent because when you sub in the formulae for i_V and I_3 you get $i_L + i_1 - i_2 + i_3 = i_1 - i_2 + i_3 + i_L$ which is true for all i_1, i_2, i_3, i_L.

The voltage between the nodes 0 and 1 is V. It is also (following the path 021) $\frac{1}{C_3}Q_3 + i_1R_1$, (following the path 041) $i_3R_3 + \frac{1}{C_1}Q_1$, (following the path 0341) $i_LR_L + \frac{1}{C_2}Q_2 + \frac{1}{C_1}Q_1$ and (following the path 0321) $i_LR_L + i_2R_2 + i_1R_1$. Hence, by Kirchhoff's voltage law

$$\begin{aligned}
V &= \tfrac{1}{C_3}Q_3 + i_1R_1 \\
V &= i_3R_3 + \tfrac{1}{C_1}Q_1 \\
V &= i_LR_L + \tfrac{1}{C_2}Q_2 + \tfrac{1}{C_1}Q_1 \\
V &= i_LR_L + i_2R_2 + i_1R_1
\end{aligned}$$

To eliminate the Q_i's, apply $\frac{d}{dt}$ to the first three equations:

$$\begin{aligned}
\tfrac{dV}{dt} &= \tfrac{1}{C_3}\tfrac{dQ_3}{dt} + R_1\tfrac{di_1}{dt} = \tfrac{1}{C_3}I_3 + R_1\tfrac{di_1}{dt} \\
\tfrac{dV}{dt} &= R_3\tfrac{di_3}{dt} + \tfrac{1}{C_1}\tfrac{dQ_1}{dt} = R_3\tfrac{di_3}{dt} + \tfrac{1}{C_1}I_1 \\
\tfrac{dV}{dt} &= R_L\tfrac{di_L}{dt} + \tfrac{1}{C_2}\tfrac{dQ_2}{dt} + \tfrac{1}{C_1}\tfrac{dQ_1}{dt} = R_L\tfrac{di_L}{dt} + \tfrac{1}{C_2}I_2 + \tfrac{1}{C_1}I_1 \\
V &= i_LR_L + i_2R_2 + i_1R_1
\end{aligned}$$

Sub in for I_1, I_2, I_3:

$$\begin{aligned}
R_1 \tfrac{di_1}{dt} &= -\tfrac{1}{C_3} i_1 + \tfrac{1}{C_3} i_2 + \tfrac{dV}{dt} \\
R_3 \tfrac{di_3}{dt} &= \tfrac{1}{C_1} i_2 - \tfrac{1}{C_1} i_3 - \tfrac{1}{C_1} i_L + \tfrac{dV}{dt} \\
R_L \tfrac{di_L}{dt} &= -\tfrac{1}{C_1} I_1 - \tfrac{1}{C_2} I_2 + \tfrac{dV}{dt} = -\tfrac{1}{C_1}(i_L - i_2 + i_3) - \tfrac{1}{C_2}(i_L - i_2) + \tfrac{dV}{dt} \\
&= \big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) i_2 - \tfrac{1}{C_1} i_3 - \big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) i_L + \tfrac{dV}{dt} \\
V &= i_L R_L + i_2 R_2 + i_1 R_1
\end{aligned}$$

The current i_2 can be eliminated using the last equation $i_2 = -\frac{R_1}{R_2} i_1 - \frac{R_L}{R_2} i_L + \frac{1}{R_2} V$.

$$\begin{aligned}
\tfrac{di_1}{dt} &= -\tfrac{1}{R_1 C_3} i_1 + \tfrac{1}{R_1 C_3}\big(-\tfrac{R_1}{R_2} i_1 - \tfrac{R_L}{R_2} i_L + \tfrac{1}{R_2} V\big) + \tfrac{1}{R_1} \tfrac{dV}{dt} \\
\tfrac{di_3}{dt} &= \tfrac{1}{R_3 C_1}\big(-\tfrac{R_1}{R_2} i_1 - \tfrac{R_L}{R_2} i_L + \tfrac{1}{R_2} V\big) - \tfrac{1}{R_3 C_1} i_3 - \tfrac{1}{R_3 C_1} i_L + \tfrac{1}{R_3} \tfrac{dV}{dt} \\
\tfrac{di_L}{dt} &= \tfrac{1}{R_L}\big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big)\big(-\tfrac{R_1}{R_2} i_1 - \tfrac{R_L}{R_2} i_L + \tfrac{1}{R_2} V\big) - \tfrac{1}{R_L C_1} i_3 - \tfrac{1}{R_L}\big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) i_L + \tfrac{1}{R_L} \tfrac{dV}{dt}
\end{aligned}$$

Finally, collect up terms

$$\begin{aligned}
\tfrac{di_1}{dt} &= -\big(\tfrac{1}{R_1 C_3} + \tfrac{1}{R_2 C_3}\big) i_1 - \tfrac{R_L}{R_1 R_2 C_3} i_L + \tfrac{1}{R_1 R_2 C_3} V + \tfrac{1}{R_1} \tfrac{dV}{dt} \\
\tfrac{di_3}{dt} &= -\tfrac{R_1}{R_2 R_3 C_1} i_1 - \tfrac{1}{R_3 C_1} i_3 - \big(\tfrac{R_L}{R_2 R_3 C_1} + \tfrac{1}{R_3 C_1}\big) i_L + \tfrac{1}{R_2 R_3 C_1} V + \tfrac{1}{R_3} \tfrac{dV}{dt} \\
\tfrac{di_L}{dt} &= -\tfrac{R_1}{R_2 R_L}\big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) i_1 - \tfrac{1}{R_L C_1} i_3 - \big(\tfrac{1}{R_2} + \tfrac{1}{R_L}\big)\big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) i_L + \tfrac{1}{R_2 R_L}\big(\tfrac{1}{C_1} + \tfrac{1}{C_2}\big) V + \tfrac{1}{R_L} \tfrac{dV}{dt}
\end{aligned}$$

Appendix IV.A Roots of Polynomials

Here are some tricks for finding roots of polynomials that work well on exams and homework assignments, where polynomials tend to have integer coefficients and lots of integer, or at least rational roots.

Trick # 1

If r or $-r$ is an integer root of a polynomial $a_n x^n + \ \cdots\ + a_1 x + a_0$ with integer coefficients, then r is a factor of the constant term a_0.

To see that this is true, just observe that for any root $\pm r$

$$a_n(\pm r)^n + \ \cdots\ + a_1(\pm r) + a_0 = 0 \quad \Longrightarrow \quad a_0 = -\big[a_n(\pm r)^n + \ \cdots\ + a_1(\pm r)\big]$$

Every term on the right hand side is an integer times a strictly positive power of r. So the right hand side, and hence the left hand side, is some integer times r.

Example IV.A.1 $P(\lambda) = \lambda^3 - \lambda^2 + 2$.

The constant term in this polynomial is $2 = 1 \times 2$. So the only candidates for integer roots are $\pm 1,\ \pm 2$. Trying each in turn

$$P(1) = 2 \qquad P(-1) = 0 \qquad P(2) = 6 \qquad P(-2) = -10$$

so the only integer root is -1.

Trick # 2

If b/d or $-b/d$ is a rational root in lowest terms (i.e. b and d are integers with no common factors) of a polynomial $a_n x^n + \ \cdots\ + a_1 x + a_0$ with integer coefficients, then the numerator b is a factor of the constant term a_0 and the denominator d is a factor of a_n.

For any root $\pm b/d$

$$a_n(\pm b/d)^n + \ \cdots\ + a_1(\pm b/d) + a_0 = 0$$

Multiply through by d^n

$$a_0 d^n = -\big[a_n(\pm b)^n + a_{n-1}d(\pm b)^{n-1} + \ \cdots\ + a_1 d^{n-1}(\pm b)\big]$$

Every term on the right hand side is an integer times a strictly positive power of b. So the right hand side is some integer times b. The left hand side is $d^n a_0$ and d does not contain any factor that is a factor of b. So a_0 must be some integer times b. Similarly, every term on the right hand side of

$$a_n(\pm b)^n = -\big[a_{n-1}d(\pm b)^{n-1} \ \cdots\ + a_1 d^{n-1}(\pm b) + a_0 d^n\big]$$

is an integer times a strictly positive power of d. So the right hand side is some integer times d. The left hand side is $a_n(\pm b)^n$ and b does not contain any factor that is a factor of d. So a_n must be some integer times d.

Example IV.A.2 $P(\lambda) = 2\lambda^2 - \lambda - 3$.

The constant term in this polynomial is $3 = 1 \times 3$ and the coefficient of the highest power of λ is $2 = 1 \times 2$. So the only candidates for integer roots are $\pm 1,\ \pm 3$ and the only candidates for fractional roots are $\pm\frac{1}{2},\ \pm\frac{3}{2}$.

$$P(1) = -2 \quad P(-1) = 0 \quad P\big(\pm 3\big) = 18 \mp 3 - 3 \neq 0 \quad P\big(\pm\tfrac{1}{2}\big) = \tfrac{1}{2} \mp \tfrac{1}{2} - 3 \neq 0 \quad P\big(\pm\tfrac{3}{2}\big) = \tfrac{9}{2} \mp \tfrac{3}{2} - 3$$

so the roots are -1 and $\frac{3}{2}$.

Trick # 3

Once you have found one root r of a polynomial, you can divide by $\lambda - r$, using the long division algorithm you learned in public school, but with 10 replaced by λ, to reduce the degree of the polynomial by one.

Example IV.A.3 $P(\lambda) = \lambda^3 - \lambda^2 + 2$.

We have already determined that -1 is a root of this polynomial. So we divide $\frac{\lambda^3-\lambda^2+2}{\lambda+1}$.

$$
\begin{array}{r|l}
 & \lambda^2 - 2\lambda + 2 \\
\cline{2-2}
\lambda+1 & \lambda^3 - \lambda^2 + \qquad 2 \\
 & \underline{\lambda^3 + \lambda^2} \\
 & \quad -2\lambda^2 \\
 & \quad \underline{-2\lambda^2 - 2\lambda} \\
 & \qquad\qquad 2\lambda + 2 \\
 & \qquad\qquad \underline{2\lambda + 2} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

The first term, λ^2, in the quotient is chosen so that when you multiply it by the denominator, $\lambda^2(\lambda+1) = \lambda^3 + \lambda^2$, the leading term, λ^3, matches the leading term in the numerator, $\lambda^3 - \lambda^2 + 2$, exactly. When you subtract $\lambda^2(\lambda+1) = \lambda^3 + \lambda^2$ from the numerator $\lambda^3 - \lambda^2 + 2$ you get the remainder $-2\lambda^2 + 2$. Just like in public school, the 2 is not normally "brought down" until it is actually needed. The next term, -2λ, in the quotient is chosen so that when you multiply it by the denominator, $-2\lambda(\lambda+1) = -2\lambda^2 - 2\lambda$, the leading term $-2\lambda^2$ matches the leading term in the remainder exactly. And so on. Note that we finally end up with a remainder 0. Since -1 is a root of the numerator, $\lambda^3 - \lambda^2 + 2$, the denominator $\lambda - (-1)$ must divide the numerator exactly.

Here is an alternative to long division that involves more writing. In the previous example, we know that $\frac{\lambda^3-\lambda^2+2}{\lambda+1}$ must be a polynomial (since -1 is a root of the numerator) of degree 2. So

$$\frac{\lambda^3 - \lambda^2 + 2}{\lambda + 1} = a\lambda^2 + b\lambda + c$$

for some, as yet unknown, coefficients a, b and c. Cross multiplying and simplifying

$$\begin{aligned}
\lambda^3 - \lambda^2 + 2 &= (a\lambda^2 + b\lambda + c)(\lambda + 1) \\
&= a\lambda^3 + (a+b)\lambda^2 + (b+c)\lambda + c
\end{aligned}$$

Matching coefficients of the various powers of λ on the left and right hand sides

$$a = 1 \qquad a + b = -1 \qquad b + c = 0 \qquad c = 2$$

forces

$$a = 1 \qquad b = -2 \qquad c = 2$$

Example IV.A.4 Suppose that we wish to find the roots of $P(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$. We start by looking for integer roots. Since $P(\lambda)$ has integer coefficients, any integer root must divide the constant term, namely 6, exactly. So the only possible candidates for integer roots are ± 1, ± 2, ± 3 and ± 6. Start by testing ± 1. (They are easier to test than the others.)

$$\begin{aligned}
P(-1) &= -(-1)^3 + 6(-1)^2 - 11(-1) + 6 = 24 \neq 0 \\
P(1) &= -(1)^3 + 6(1)^2 - 11(1) + 6 = 0
\end{aligned}$$

So -1 is not a root, but 1 is. Consequently $(\lambda - 1)$ must be a factor in $P(\lambda)$. Rather than test the other candidates, it is usually more efficient to factor $P(\lambda)$, using the knowledge that $P(\lambda)$ must be of the form

$$P(\lambda) = (\lambda - 1)\big(a\lambda^2 + b\lambda + c\big)$$

When we multiply out the right hand side the λ^3 term will be $a\lambda^3$. Since the λ^3 term in $P(\lambda)$ is $-\lambda^3$, we must have $a = -1$. Similarly, when we multiply out the right hand side the constant term will be $-c$. Since the constant term in $P(\lambda)$ is 6, we must have $c = -6$. That only leaves b, which we determine by observing that when we multiply out the right hand side the λ^2 term will be $(b-a)\lambda^2$. Since the λ^2 term in $P(\lambda)$ is $6\lambda^2$, we must have $b - a = 6$ and hence $b = 6 + a = 5$. Thus

$$P(\lambda) = (\lambda - 1)\big(-\lambda^2 + 5\lambda - 6\big) = -(\lambda - 1)\big(\lambda^2 - 5\lambda + 6\big) = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

So the roots are 1, 2 and 3.

Appendix B. Complex Numbers

A complex number is nothing more than a point in the xy–plane. The first component, x, of the complex number (x, y) is called its real part and the second component, y, is called its imaginary part, even though there is nothing imaginary about it. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) are defined by

$$
\begin{aligned}
(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\
(x_1, y_1)\,(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)
\end{aligned}
$$

respectively. We'll get an effective memory aid for the definition of multiplication shortly. It is conventional to use the notation $x + iy$ (or in electrical engineering country $x + jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, The sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2)\\
z_1z_2 &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)
\end{aligned}
$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$
\begin{aligned}
z_1 + z_2 &= z_2 + z_1 & z_1z_2 &= z_2z_1\\
z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2z_3) &= (z_1z_2)z_3\\
0 + z_1 &= z_1 & 1z_1 &= z_1\\
z_1(z_2 + z_3) &= z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3
\end{aligned}
$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$. The inverse, z^{-1} or $\frac{1}{z}$, of any complex number $z = x + iy$, other than 0, is defined by $\frac{1}{z}z = 1$. We shall see below that it is given by the formula $\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$. The complex number i has the special property

$$
i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1
$$

To remember how to multiply complex numbers, you just have to supplement the familiar rules of the real number system with $i^2 = -1$.

The absolute value, or modulus, $|z|$ of $z = x + iy$ is given by

$$
|z| \;=\; \sqrt{x^2 + y^2}
$$

It is just the distance between z, viewed as a point in the xy–plane, and the origin. We have

$$
\begin{aligned}
|z_1z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}\\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2}\\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2}\\
&= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}\\
&= |z_1||z_2|
\end{aligned}
$$

for all complex numbers z_1 and z_2.

The complex conjugate $\bar{z}$ of the complex number $z = x + iy$ is defined by $\bar{z} = x - iy$. That is, to take the complex conjugate, you just replace every i by $-i$. Note that

$$
z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2
$$

Thus

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

for all $z \neq 0$. This is the formula for $\frac{1}{z}$ given above.

The complex conjugate is useful in simplifying ratios of complex numbers like $\frac{2+i}{1-i}$. Just multiply both the numerator and denominator by the complex conjugate of the denominator.

$$\frac{2+i}{1-i} = \frac{2+i}{1-i}\,\frac{1+i}{1+i} = \frac{2+2i+i-1}{1+i-i+1} = \frac{1+3i}{2}$$

As it had to be, the denominator is now real. The complex conjugate is also useful for extracting real and imaginary parts of a complex number. If $z = x + iy$ is any complex number

$$\begin{aligned} x &= \tfrac{1}{2}(z + \bar{z}) \\ y &= \tfrac{1}{2i}(z - \bar{z}) \end{aligned}$$

Appendix C. The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z, is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned}
e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}\left\{(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \cos y_2 \sin y_1)\right\} \\
&= e^{x_1+x_2}\left\{\cos(y_1 + y_2) + i\sin(y_1 + y_2)\right\} \\
&= e^{(x_1+x_2)+i(y_1+y_2)} \\
&= e^{z_1+z_2}
\end{aligned}$$

so that the familiar multiplication formula for real exponentials also applies to complex exponentials. For any complex number $c = \alpha + i\beta$ and real number t

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned}
\tfrac{d}{dt}e^{ct} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)] \\
&= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\
&= ce^{ct}
\end{aligned}$$

is also the familiar one.

Relationship with sin and cos. When θ is a real number

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{-i\theta} &= \cos\theta - i\sin\theta
\end{aligned}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

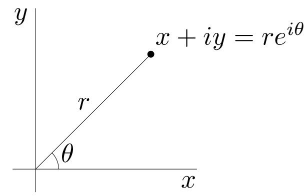
$$\begin{aligned}
\cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \\
\sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})
\end{aligned}$$

These formulae make it easy to derive trig identities. For example

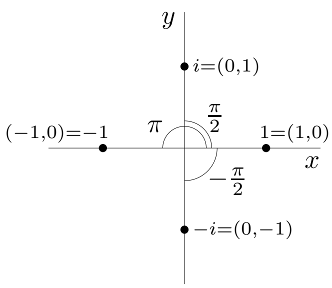
$$\begin{aligned}
\cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\
&= \tfrac{1}{4}\left[e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}\right] \\
&= \tfrac{1}{4}\left[e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}\right] \\
&= \tfrac{1}{2}\big[\cos(\theta+\phi) + \cos(\theta-\phi)\big] \\
\cos^4\theta &= \tfrac{1}{2^4}\left[e^{i\theta} + e^{-i\theta}\right]^4 \\
&= \tfrac{1}{2^4}\left[e^{i4\theta} + 4e^{i3\theta}e^{-i\theta} + 6e^{i2\theta}e^{-i2\theta} + 4e^{i\theta}e^{-i3\theta} + e^{-4i\theta}\right] \\
&= \tfrac{1}{16}\left[e^{i4\theta} + e^{-4i\theta} + 4e^{i2\theta} + 4e^{-i2\theta} + 6\right] \\
&= \tfrac{1}{8}\cos(4\theta) + \tfrac{1}{2}\cos(2\theta) + \tfrac{3}{8}
\end{aligned}$$

Polar Coordinates. Let $z = x + iy$ be any complex number. Writing x and y in polar coordinates in the usual way gives

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



In particular, $1 = 1 + i0$ has $r = 1$ and $\theta = 0$ or 2π or, in general, $\theta = 2k\pi$ for any integer k. So, $1 = re^{i\theta} = e^{2k\pi i}$ for any integer k. Similarly, $-1 = -1 + i0$ has $r = 1$ and $\theta = \pi$ or, in general, $\theta = \pi + 2k\pi$ for any integer k, so that $-1 = e^{(\pi+2k\pi)i}$ for any integer k.



$$\begin{array}{rcccccl} 1 &=& e^{i0} &=& e^{2\pi i} &=& e^{2k\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -1 &=& e^{i\pi} &=& e^{3\pi i} &=& e^{(1+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ i &=& e^{i\pi/2} &=& e^{\frac{5}{2}\pi i} &=& e^{(\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -i &=& e^{-i\pi/2} &=& e^{\frac{3}{2}\pi i} &=& e^{(-\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \end{array}$$

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The n^{th} roots of unity are, by definition, all solutions z of

$$z^n \;=\; 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} \;=\; 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy–plane (i.e. $re^{i\theta} = r'e^{i\theta'}$) if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The n^{th} roots of unity are all complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi i\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The n distinct n^{th} roots of unity are

$$1\ ,\ e^{2\pi i\frac{1}{n}}\ ,\ e^{2\pi i\frac{2}{n}}\ ,\ e^{2\pi i\frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i\frac{n-1}{n}}$$