Example II.3 (Feldman's notes)

$$
\begin{aligned}
2x_1 + \ x_2 + 3x_3 &= \ \ 1 \\
4x_1 + 5x_2 + 7x_3 &= \ \ 7 \\
2x_1 - 5x_2 + 5x_3 &= -7
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c|c} 2 & 1 & 3 & 1 & 7 \\ 4 & 5 & 7 & 7 & 23 \\ 2 & -5 & 5 & -7 & -5 \end{array}\right]
$$

$$
\begin{array}{c}
(1) \\ (2) - 2(1) \\ (3) - (1)
\end{array}
\left[\begin{array}{ccc|c|c} 2 & 1 & 3 & 1 & 7 \\ 0 & 3 & 1 & 5 & 9 \\ 0 & -6 & 2 & -8 & -12 \end{array}\right]
$$

$$
\begin{array}{c}
(1) \\ (2) \\ (3) + 2(2)
\end{array}
\left[\begin{array}{ccc|c|c} 2 & 1 & 3 & 1 & 7 \\ 0 & 3 & 1 & 5 & 9 \\ 0 & 0 & 4 & 2 & 6 \end{array}\right]
$$

$$
\begin{aligned}
(3) \quad &\Longrightarrow \quad 4x_3 = 2 \quad &&\Longrightarrow \quad x_3 = \tfrac{1}{2} \\
(2) \quad &\Longrightarrow \quad 3x_2 + \tfrac{1}{2} = 5 \quad &&\Longrightarrow \quad x_2 = \tfrac{3}{2} \\
(1) \quad &\Longrightarrow \quad 2x_1 + \tfrac{3}{2} + 3 \times \tfrac{1}{2} = 1 \quad &&\Longrightarrow \quad x_1 = -1
\end{aligned}
$$

Check

$$
\begin{aligned}
2(-1) + \quad \tfrac{3}{2} + 3 \times \tfrac{1}{2} &= \ \ 1 \\
4(-1) + 5 \times \tfrac{3}{2} + 7 \times \tfrac{1}{2} &= \ \ 7 \\
2(-1) - 5 \times \tfrac{3}{2} + 5 \times \tfrac{1}{2} &= -7
\end{aligned}
$$

Example II.4 (Feldman's notes)

$$
\begin{aligned}
x_1 + 2x_2 + \ x_3 + 2x_4 + \ x_5 &= 1 \\
2x_1 + 4x_2 + 4x_3 + 6x_4 + \ x_5 &= 2 \\
3x_1 + 6x_2 + \ x_3 + 4x_4 + 5x_5 &= 4 \\
x_1 + 2x_2 + 3x_3 + 5x_4 + \ x_5 &= 4
\end{aligned}
$$

$$
\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 4 & 4 & 6 & 1 & 2 \\ 3 & 6 & 1 & 4 & 5 & 4 \\ 1 & 2 & 3 & 5 & 1 & 4 \end{array}\right] \begin{array}{c} 8 \\ 19 \\ 23 \\ 16 \end{array}
$$

$$
\begin{array}{c} (1) \\ (2)-2(1) \\ (3)-3(1) \\ (4)-(1) \end{array}
\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & -2 & -2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 0 & 3 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ -1 \\ 8 \end{array}
$$

$$
\begin{array}{c} (1) \\ (2) \\ (3)+(2) \\ (4)-(2) \end{array}
\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ 2 \\ 5 \end{array}
$$

$$
\begin{array}{c} (1) \\ (2) \\ (4) \\ (3) \end{array}
\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ 5 \\ 2 \end{array}
$$

Back Solution - the direct method

$$
\begin{array}{lrcl}
(4) \Rightarrow & x_5 = 1 & & \\
(3) \Rightarrow & x_4 + 1 = 3 & \Rightarrow & x_4 = 2 \\
(2) \Rightarrow & 2x_3 + 2 \times 2 - 1 = 0 & \Rightarrow & x_3 = -\frac{3}{2} \\
(1) \Rightarrow & x_1 + 2x_2 - \frac{3}{2} + 2 \times 2 + 1 = 1 & \Rightarrow & x_1 + 2x_2 = -\frac{5}{2} \\
 & & \Rightarrow & x_2 = t, \text{ arbitrary} \\
 & & & x_1 = -\frac{5}{2} - 2t
\end{array}
$$

Back Solution - the row reduction method

$$
\begin{array}{r}
(1) - (4) \\
(2) + (4) \\
(3) - (4) \\
(4)
\end{array}
\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{c}
6 \\ 5 \\ 3 \\ 2
\end{array}
$$

$$
\begin{array}{r}
(1) - 2(3) \\
(2) - 2(3) \\
(3) \\
(4)
\end{array}
\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 0 & 0 & -4 \\
0 & 0 & 2 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{c}
0 \\ -1 \\ 3 \\ 2
\end{array}
$$

$$
\begin{array}{r}
(1) \\
(2)/2 \\
(3) \\
(4)
\end{array}
\left[\begin{array}{ccccc|c}
1 & 2 & 1 & 0 & 0 & -4 \\
0 & 0 & 1 & 0 & 0 & -\frac{3}{2} \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{c}
0 \\ -\frac{1}{2} \\ 3 \\ 2
\end{array}
$$

$$
\begin{array}{r}
(1) - (2) \\
(2) \\
(3) \\
(4)
\end{array}
\left[\begin{array}{ccccc|c}
1 & 2 & 0 & 0 & 0 & -5/2 \\
0 & 0 & 1 & 0 & 0 & -3/2 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{l}
\Rightarrow x_2 = t, \; x_1 = -\frac{5}{2} - 2t \\
\Rightarrow x_3 = -\frac{3}{2} \\
\Rightarrow x_4 = 2 \\
\Rightarrow x_5 = 1
\end{array}
$$

Example II.5 (Feldman's notes) – Check

$$\begin{aligned}
\big(-\tfrac{5}{2}-2t\big)+2t- \quad \tfrac{3}{2} \quad +2\times 2+1&=1\\
2\big(-\tfrac{5}{2}-2t\big)+4t+4\big(-\tfrac{3}{2}\big)+6\times 2+1&=2\\
3\big(-\tfrac{5}{2}-2t\big)+6t- \quad \tfrac{3}{2} \quad +4\times 2+5&=4\\
\big(-\tfrac{5}{2}-2t\big)+2t+3\big(-\tfrac{3}{2}\big)+5\times 2+1&=4
\end{aligned}$$

$$\begin{aligned}
(-\tfrac{5}{2}-\tfrac{3}{2}+2\times 2+1)+ \quad (-2+2)t&=1 \qquad (1')\\
(-2\times\tfrac{5}{2}-4\times\tfrac{3}{2}+6\times 2+1)+(-2\times 2+4)t&=2 \qquad (2')\\
(-3\times\tfrac{5}{2}-\tfrac{3}{2}+4\times 2+5)+(-3\times 2+6)t&=4 \qquad (3')\\
(-\tfrac{5}{2}-3\times\tfrac{3}{2}+5\times 2+1)+ \quad (-2+2)t&=4 \qquad (4')
\end{aligned}$$

$$\begin{aligned}
(1)+(0)t&=1 \qquad (1')\\
(2)+(0)t&=2 \qquad (2')\\
(4)+(0)t&=4 \qquad (3')\\
(4)+(0)t&=4 \qquad (4')
\end{aligned}$$