**Example III.19** (Feldman's notes)

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. We look for a matrix $B$ that obeys $AB = I$. The columns of $I$ are $\widehat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\widehat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Denote the columns of $B$ by $\vec{B}_1$ and $\vec{B}_2$. That is

$$B = \begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix} = \big[\vec{B}_1\ \vec{B}_2\big] \qquad \vec{B}_1 = \begin{bmatrix} X \\ Y \end{bmatrix}\ \vec{B}_2 = \begin{bmatrix} X' \\ Y' \end{bmatrix}$$

We are to find a $B$ obeying

$$AB = \begin{bmatrix} X+Y & X'+Y' \\ X+2Y & X'+2Y' \end{bmatrix} = \big[A\vec{B}_1\ A\vec{B}_2\big] = \big[\widehat{\mathbf{e}}_1\ \widehat{\mathbf{e}}_2\big]$$

or, equivalently, $A\vec{B}_1 = \widehat{\mathbf{e}}_1$, $A\vec{B}_2 = \widehat{\mathbf{e}}_2$. We wish to solve for $\vec{B}_1$ and $\vec{B}_2$. The augmented matrices for these two systems of equations are $\big[A\big|\widehat{\mathbf{e}}_1\big]$ and $\big[A\big|\widehat{\mathbf{e}}_2\big]$. For efficiency, we combine them into a single larger agumented matrix

$$\big[A\,\big|\,\widehat{\mathbf{e}}_1\ \widehat{\mathbf{e}}_2\big] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

We can apply row operations to bring this to reduced row echelon form

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} \\ (2)-(1) \end{matrix} \qquad\qquad \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{matrix} (1)-(2) \\ \\ \end{matrix}$$

The augmented matrix on the right represents the two systems of equations $I\vec{B}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $I\vec{B}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus

$$\vec{B}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad B = \big[\vec{B}_1\ \vec{B}_2\big] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$