MATHEMATICS 200 December 2005 Final Exam

1. One side of a right triangle is measured to be 3 with a maximum possible error of ± 0.1, and the other side is measured to be 4 with a maximum possible error of ± 0.2. Use the linear approximation to estimate the maximum possible error in calculating the length of the hypotenuse of the right triangle.

2. Assume that $f(x, y)$ satisfies Laplaces equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. Show that this is also the case for the composite function $g(s, t) = f(s - t, s + t)$. That is, show that $\frac{\partial^2 g}{\partial s^2} + \frac{\partial^2 g}{\partial t^2} = 0$. You may assume that $f(x, y)$ is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partials derivatives apply.

3. You are standing at a location where the surface of the earth is smooth. The slope in the southern direction is 4 and the slope in the south–eastern direction is $\sqrt{2}$. Find the slope in the eastern direction.

4. Consider the surface $z = f(x, y)$ defined implicitly by the equation $xyz^2 + y^2z^3 = 3 + x^2$. Use a 3–dimensional gradient vector to find the equation of the tangent plane to this surface at the point $(-1, 1, 2)$. Write your answer in the form $z = ax + by + c$, where a, b and c are constants.

5. Let $z = f(x, y) = (y^2 - x^2)^2$.

   (a) Make a reasonably accurate sketch of the level curves in the xy–plane of $z = f(x, y)$ for $z = 0$, 1 and 16. Be sure to show the units on the coordinate axes.

   (b) Verify that $(0, 0)$ is a critical point for $z = f(x, y)$, and determine from part (a) or directly from the formula for $f(x, y)$ whether $(0, 0)$ is a local minimum, a local maximum or a saddle point.

   (c) Can you use the Second Derivative Test to determine whether the critical point $(0, 0)$ is a local minimum, a local maximum or a saddle point? Give reasons for your answer.

6. Use the Method of Lagrange Multipliers to find the minimum value of $z = x^2 + y^2$ subject to $x^2y = 1$. At which point or points does the minimum occur?

7. Find the centre of mass of the region D in the xy–plane defined by the inequalities $x^2 \le y \le 1$, assuming that the mass density function is given by $\rho(x, y) = y$.

8. Consider the region E in 3–dimensions specified by the inequalities $x^2 + y^2 \le 2y$ and $0 \le z \le \sqrt{x^2 + y^2}$.

   (a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinate axes.

   (b) Use polar coordinates to find the volume of E. Note that you will be "using polar coordinates" if you solve this problem by means of cylindrical coordinates.
   Hint: $\int \sin^n u \text{ d}u = -\frac{1}{n} \sin^{n-1} u \ \cos u + \frac{n-1}{n} \int \sin^{n-2} u \text{ d}y$

9. A triple integral $\iiint_E f \, \mathrm{d}V$ is given in iterated form by

$$\int_{y=-1}^{y=1} \int_{z=0}^{z=1-y^2} \int_{x=0}^{2-y-z} f(x,y,z) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y$$

(a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinate axes.

(b) Rewrite the triple integral $\iiint_E f \, \mathrm{d}V$ as one or more iterated triple integrals in the order

$$\int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x,y,z) \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$$