MATHEMATICS 200 December 2006 Final Exam

1. Consider the surface given by:

$$z^3 - xyz^2 - 4x = 0.$$

   (a) Find expressions for $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ as functions of x, y, z.

   (b) Evaluate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1, 1, 2)$.

   (c) Measurements are made with errors, so that $x = 1 \pm 0.03$ and $y = 1 \pm 0.02$. Find the corresponding maximum error in measuring z.

   (d) A particle moves over the surface along the path whose projection in the xy–plane is given in terms of the angle θ as

$$x(\theta) = 1 + \cos\theta,\ \ y(\theta) = \sin\theta$$

   from the point $A : x = 2,\ y = 0$ to the point $B : x = 1,\ y = 1$. Find $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ at points A and B.

2. A hiker is walking on a mountain with height above the $z = 0$ plane given by

$$z = f(x, y) = 6 - xy^2$$

   The positive x–axis points east and the positive y–axis points north, and the hiker starts from the point $P(2, 1, 4)$.

   (a) In what direction should the hiker proceed from P to ascend along the steepest path? What is the slope of the path?

   (b) Walking north from P, will the hiker start to ascend or descend? What is the slope?

   (c) In what direction should the hiker walk from P to remain at the same height?

3. (a) Find and classify all critical points of the function

$$f(x, y) = x^3 - y^3 - 2xy + 6.$$

   (b) Use the method of Lagrange Multipliers to find the maximum and minimum values of

$$f(x, y) = xy$$

   subject to the constraint

$$x^2 + 2y^2 = 1.$$

4. The integral I is defined as

$$I = \iint_R f(x, y)\ \mathrm{d}A = \int_1^{\sqrt{2}} \int_{1/y}^{\sqrt{y}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\sqrt{2}}^{4} \int_{y/2}^{\sqrt{y}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

(a) Sketch the region R.

(b) Re–write the integral I by reversing the order of integration.

(c) Compute the integral I when $f(x, y) = x/y$.

5. (a) Sketch the region $\mathcal{L}$ (in the first quadrant of the xy–plane) with boundary curves

$$x^2 + y^2 = 2,\ x^2 + y^2 = 4,\ y = x,\ y = 0.$$

The mass of a thin lamina with a density function $\rho(x, y)$ over the region $\mathcal{L}$ is given by

$$M = \iint_{\mathcal{L}} \rho(x, y)\,\mathrm{d}A$$

(b) Find an expression for M as an integral in polar coordinates.

(c) Find M when

$$\rho(x, y) = \frac{2xy}{x^2 + y^2}$$

6. (a) A triple integral $\iiint_E f(x, y, z)\ \mathrm{d}V$ is given in the iterated form

$$J = \int_0^1 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4z} f(x, y, z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

(i) Sketch the domain E in 3–dimensions. Be sure to show the units.

(ii) Rewrite the integral as one or more iterated integrals in the form

$$J = \int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x, y, z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

(b) Use spherical coordinates to evaluate the integral

$$I = \iiint_D z\ \mathrm{d}V$$

where D is the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2+y^2+z^2 = 4$.