MATHEMATICS 200 December 2008 Final Exam

1. A surface is given by
$$z = x^2 - 2xy + y^2.$$
   (a) Find the equation of the tangent plane to the surface at $x = a$, $y = 2a$.

   (b) For what value of a is the tangent plane parallel to the plane $x - y + z = 1$?

2. The pressure in a solid is given by
$$P(s, r) = sr(4s^2 - r^2 - 2)$$
where s is the specific heat and r is the density. We expect to measure (s, r) to be approximately $(2, 2)$ and would like to have the most accurate value for P. There are two different ways to measure s and r. Method 1 has an error in s of ± 0.01 and an error in r of ± 0.1, while method 2 has an error of ± 0.02 for both s and r.

   Should we use method 1 or method 2? Explain your reasoning carefully.

3. $u(x, y)$ is defined as
$$u(x, y) = e^y F\left(xe^{-y^2}\right)$$
for an arbitrary function $F(z)$.

   (a) If $F(z) = \ln(z)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

   (b) For an arbitrary $F(z)$ show that $u(x, y)$ satisfies
$$2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

4. The air temperature $T(x, y, z)$ at a location (x, y, z) is given by:
$$T(x, y, z) = 1 + x^2 + yz.$$
   (a) A bird passes through $(2, 1, 3)$ travelling towards $(4, 3, 4)$ with speed 2. At what rate does the air temperature it experiences change at this instant?

   (b) If instead the bird maintains constant altitude ($z = 3$) as it passes through $(2, 1, 3)$ while also keeping at a fixed air temperature, $T = 8$, what are its two possible directions of travel?

5. (a) Find all saddle points, local minima and local maxima of the function
$$f(x, y) = x^3 + x^2 - 2xy + y^2 - x.$$
   (b) Use Lagrange multipliers to find the points on the sphere $z^2 + x^2 + y^2 - 2y - 10 = 0$ closest to and farthest from the point $(1, -2, 1)$.

6. Consider the integral

$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\, \mathrm{d}x\, \mathrm{d}y$$

   (a) Sketch the region of integration.

   (b) Evaluate I.

7. Let R be the region bounded on the left by $x = 1$ and on the right by $x^2 + y^2 = 4$. The density in R is

$$\rho(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

   (a) Sketch the region R.

   (b) Find the mass of R.

   (c) Find the centre-of-mass of R.

   Note: You may use the result $\int \sec(\theta)\ \mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.

8. Let

$$I = \iiint_T xz\ \mathrm{d}V$$

   where T is the eighth of the sphere $x^2 + y^2 + z^2 \leq 1$ with $x, y, z \geq 0$.

   (a) Sketch the volume T.

   (b) Express I as a triple integral in spherical coordinates.

   (c) Evaluate I by any method.