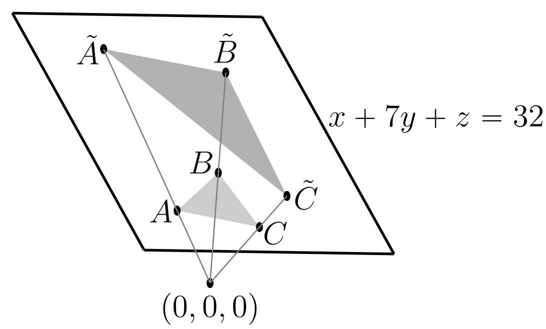
MATHEMATICS 200 December 2016 Final Exam Solutions

1. Let $A = (0, 2, 2)$, $B = (2, 2, 2)$, $C = (5, 2, 1)$.

   (a) Find the parametric equations for the line which contains A and is perpendicular to the triangle ABC.

   (b) Find the equation of the set of all points P such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{PB}$. This set forms a Plane/Line/Sphere/Cone/Paraboloid/Hyperboloid (circle one) in space.

   (c) A light source at the origin shines on the triangle ABC making a shadow on the plane $x + 7y + z = 32$. (See the diagram.) Find $\tilde{A}$.



Solution. (a) We are given one point on the line, so we just need a direction vector. That direction vector has to be perpendicular to the triangle ABC.

The fast way to get a direction vector is to observe that all three points A, B and C, and consequently the entire triangle ABC, are contained in the plane $y = 2$. A normal vector to that plane, and consequently a direction vector for the desired line, is $\hat{\boldsymbol{\jmath}}$.

Here is another, more mechanical, way to get a direction vector. The vector from A to B is $\langle 2-0\,,\, 2-2\,,\, 2-2\rangle = \langle 2, , 0, 0\rangle$ and the vector from A to C is $\langle 5-0\,,\, 2-2\,,\, 1-2\rangle = \langle 5, , 0, -1\rangle$. So a vector perpendicular to the triangle ABC is

$$\langle 2, , 0, 0\rangle \times \langle 5, , 0, -1\rangle = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 2 & 0 & 0 \\ 5 & 0 & -1 \end{bmatrix} = \langle 0\,,\, 2\,,\, 0\rangle$$

The vector $\frac{1}{2}\langle 0\,,\, 2\,,\, 0\rangle = \langle 0\,,\, 1\,,\, 0\rangle$ is also perpendicular to the triangle ABC.

So the specified line has to contain the point $(0, 2, 2)$ and have direction vector $\langle 0, 1, 0\rangle$. The parametric equations

$$\langle x, y, z\rangle = \langle 0, 2, 2\rangle + t\,\langle 0, 1, 0\rangle$$

or

$$x = 0,\ \ y = 2 + t,\ \ z = 2$$

do the job.

(b) Let P be the point (x, y, z). Then the vector from P to A is $\langle 0-x\,,\,2-y\,,\,2-z\rangle$ and the vector from P to B is $\langle 2-x\,,\,2-y\,,\,2-z\rangle$. These two vector are perpendicular if and only if

$$\begin{aligned}0 &= \langle -x\,,\,2-y\,,\,2-z\rangle \cdot \langle 2-x\,,\,2-y\,,\,2-z\rangle = x(x-2)+(y-2)^2+(z-2)^2\\ &= (x-1)^2-1+(y-2)^2+(z-2)^2\end{aligned}$$

This is a sphere.

(c) The light ray that forms $\tilde{A}$ starts at the origin, passes through A and then intersects the plane $x+7y+z=32$ at $\tilde{A}$. The line from the origin through A has vector parametric equation

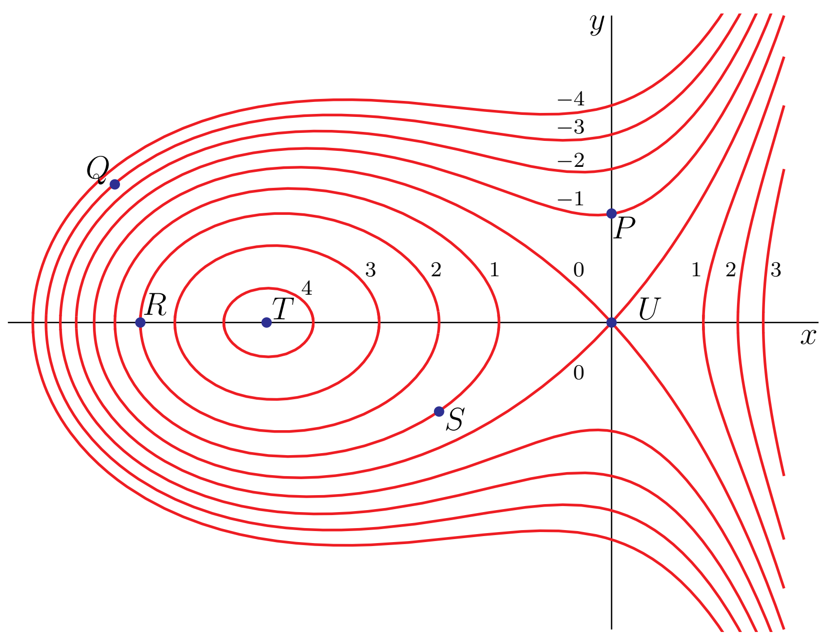
$$\langle x, y, z\rangle = \langle 0,0,0\rangle + t\,\langle 0,2,2\rangle = \langle 0,2t,2t\rangle$$

This line intersects the plane $x+7y+z=32$ at the point whose value of t obeys

$$(0)+7\,\overbrace{(2t)}^{y}+\overbrace{(2t)}^{z}=32 \iff t=2$$

So $\tilde{A}$ is $(0,4,4)$.

2. (a) Some level curves of a function $f(x, y)$ are plotted in the xy–plane below.



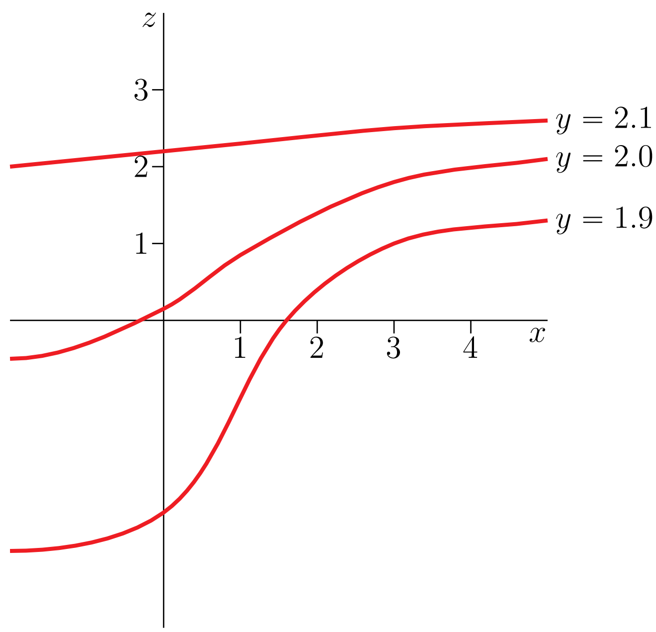
For each of the four statmements below, circle the letters of **all** points in the diagram where the situation applies. For example, if the statement were “These points are on the

y–axis", you would circle both P and U, but none of the other letters. You may assume that a local maximum occurs at point T.

(i)	∇f is zero	P R S T U
(ii)	f has a saddle point	P R S T U
(iii)	the partial derivative f_y is positive	P R S T U
(iv)	the directional derivative of f in the direction $\langle 0, -1\rangle$ is negative	P R S T U

(b) The diagram below shows three "y traces" of a graph $z = F(x, y)$ plotted on xz–axes. (Namely the intersections of the surface $z = F(x, y)$ with the three planes ($y = 1.9$, $y = 2$, $y = 2.1$). For each statement below, circle the correct word.

(i)	the first order partial derivative $F_x(1, 2)$ is	positive/negative/zero (circle one)
(ii)	F has a critical point at $(2, 2)$	true/false (circle one)
(iii)	the second order partial derivative $F_{xy}(1, 2)$ is	positive/negative/zero (circle one)



Solution. (a) (i) ∇f is zero at critical points. The point T is a local maximum and the point U is a saddle point. The remaining points P, R, S, are not critical points.

(a) (ii) Only U is a saddle point.

(a) (iii) We have $f_y(x, y) > 0$ if f increases as you move vertically upward through (x, y). Looking at the diagram, we see

$$f_y(P) < 0 \qquad f_y(Q) < 0 \qquad f_y(R) = 0 \qquad f_y(S) > 0 \qquad f_y(T) = 0 \qquad f_y(U) = 0$$

So only S works.

(a) (iv) The directional derivative of f in the direction $\langle 0, -1\rangle$ is $\nabla f \cdot \langle 0, -1\rangle = -f_y$. It is negative if and only if $f_y > 0$. So, again, only S works.

(b) (i) The function $z = F(x, 2)$ is increasing at $x = 1$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 1$. So $F_x(1, 2) > 0$.

(b) (ii) The function $z = F(x, 2)$ is also increasing (though slowly) at $x = 2$, because the $y = 2.0$ graph in the diagram has positive slope at $x = 2$. So $F_x(2, 2) > 0$. So F does *not* have a critical point at $(2, 2)$.

(b) (iii) From the diagram the looks like $F_x(1, 1.9) > F_x(1, 2.0) > F_x(1, 2.1)$. That is, it looks like the slope of the $y = 1.9$ graph at $x = 1$ is larger than the slope of the $y = 2.0$ graph at $x = 1$, which in turn is larger than the slope of the $y = 2.1$ graph at $x = 1$. So it looks like $F_x(1, y)$ decreases as y increases through $y = 2$, and consequently $F_{xy}(1, 2) < 0$.

3. Consider the functions $F(x, y, z) = z^3 + xy^2 + xz$ and $G(x, y, z) = 3x - y + 4z$. You are standing at the point $P(0, 1, 2)$.

   (a) You jump from P to $Q(0.1\,,\,0.9\,,\,1.8)$. Use the linear approximation to determine approximately the amount by which F changes.

   (b) You jump from P in the direction along which G increases most rapidly. Will F increase or decrease?

   (c) You jump from P in a direction $\langle a\,,\,b\,,\,c\rangle$ along which the rates of change of F and G are both zero. Give an example of such a direction (need not be a unit vector).

Solution. We are going to need the gradients of both F and G at $(0, 1, 2)$. So we compute

$$\frac{\partial F}{\partial x}(x, y, z) = y^2 + z \qquad \frac{\partial F}{\partial y}(x, y, z) = 2xy \qquad \frac{\partial F}{\partial z}(x, y, z) = 3z^2 + x$$
$$\frac{\partial G}{\partial x}(x, y, z) = 3 \qquad \frac{\partial G}{\partial y}(x, y, z) = -1 \qquad \frac{\partial G}{\partial z}(x, y, z) = 4$$

and then

$$\nabla F(0, 1, 2) = \langle 3, 0, 12\rangle \qquad \nabla G(0, 1, 2) = \langle 3, -1, 4\rangle$$

(a) The linear approximation to F at $(0, 1, 2)$ is

$$\begin{aligned} F(x, y, z) &\approx F(0, 1, 2) + F_x(0, 1, 2)\,x + F_y(0, 1, 2)\,(y - 1) + F_z(0, 1, 2)\,(z - 2) \\ &= 8 + 3x + 12(z - 2) \end{aligned}$$

In particular

$$F(0.1\,,\,0.9\,,\,1.8) - F(0, 1, 2) \approx 3(0.1) + 12(-0.2) = -2.1$$

(b) The direction along which G increases most rapidly at P is $\nabla G(0,1,2) = \langle 3,-1,4\rangle$. The directional derivative of F in that direction is

$$D_{\frac{\langle 3,-1,4\rangle}{\sqrt{26}}} F(0,1,2) = \nabla F(0,1,2) \cdot \frac{\langle 3,-1,4\rangle}{\sqrt{26}} = \langle 3,0,12\rangle \cdot \frac{\langle 3,-1,4\rangle}{\sqrt{26}} > 0$$

So F increases.

(c) For the rate of change of F to be zero, $\langle a\,,\, b\,,\, c\rangle$ must be perpendicular to $\nabla F(0,1,2) = \langle 3,0,12\rangle$.

For the rate of change of G to be zero, $\langle a\,,\, b\,,\, c\rangle$ must be perpendicular to $\nabla G(0,1,2) = \langle 3,-1,4\rangle$.

So any nonzero constant times

$$\det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 3 & 0 & 12 \\ 3 & -1 & 4 \end{bmatrix} = \langle 12\,,\, 24\,,\, -3\rangle = 3\,\langle 4\,,\, 8\,,\, -1\rangle$$

is an allowed direction.

4. Suppose that $f(x,y)$ is twice differentiable (with $f_{xy} = f_{yx}$), and $x = r\cos\theta$ and $y = r\sin\theta$.

   (a) Evaluate f_θ, f_r and $f_{r\theta}$ in terms of r, θ and partial derivatives of f with respect to x and y.

   (b) Let $g(x,y)$ be another function satisfying $g_x = f_y$ and $g_y = -f_x$. Express f_r and f_θ in terms of r, θ and g_r, g_θ.

Solution. (a) By the chain rule

$$\begin{aligned} \frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\, r\sin\theta\big)\big] &= -r\sin\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) + r\cos\theta\, f_y\big(r\cos\theta\,,\, r\sin\theta\big) \\ \frac{\partial}{\partial r}\big[f\big(r\cos\theta\,,\, r\sin\theta\big)\big] &= \cos\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) + \sin\theta\, f_y\big(r\cos\theta\,,\, r\sin\theta\big) \end{aligned}$$

To evaluate $\frac{\partial^2}{\partial r\,\partial\theta}\big[f\big(r\cos\theta\,,\, r\sin\theta\big)\big]$ we apply $\frac{\partial}{\partial r}$ to the first row. By the product rule followed by the chain rule, applying $\frac{\partial}{\partial r}$ to the term $-r\sin\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big)$ in the first row gives

$$\begin{aligned} &\frac{\partial}{\partial r}\Big[-r\sin\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\ &\qquad = -\sin\theta\, f_x\big(r\cos\theta\,,\, r\sin\theta\big) - r\sin\theta\,\frac{\partial}{\partial r}\Big[f_x\big(r\cos\theta\,,\, r\sin\theta\big)\Big] \\ &\qquad = -\sin\theta\, f_x - r\sin\theta\cos\theta\, f_{xx} - r\sin^2\theta\, f_{xy} \end{aligned}$$

with the arguments of f_x, f_{xx} and f_{xy} all being $\big(r\cos\theta\,,\,r\sin\theta\big)$. Similarly, applying $\frac{\partial}{\partial r}$ to the term $r\cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big)$ in $\frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big]$ gives

$$\begin{aligned}
&\frac{\partial}{\partial r}\Big[r\cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big)\Big]\\
&\qquad\qquad = \cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big) + r\cos\theta\,\frac{\partial}{\partial r}\Big[f_y\big(r\cos\theta\,,\,r\sin\theta\big)\Big]\\
&\qquad\qquad = \cos\theta\, f_y + r\,\cos^2\theta f_{yx} + r\cos\theta\sin\theta\, f_{yy}\\
&\qquad\qquad = \cos\theta\, f_y + r\,\cos^2\theta f_{xy} + r\sin\theta\cos\theta\, f_{yy}
\end{aligned}$$

since $f_{yx} = f_{xy}$. All together,

$$\begin{aligned}
\frac{\partial^2}{\partial r\,\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= \frac{\partial}{\partial r}\Big[-r\sin\theta\, f_x\big(r\cos\theta\,,\,r\sin\theta\big) + r\cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big)\Big]\\
&= -\sin\theta\, f_x + \cos\theta\, f_y\\
&\quad\ - r\sin\theta\cos\theta\, f_{xx} + r[\cos^2\theta - \sin^2\theta]\, f_{xy} + r\sin\theta\cos\theta\, f_{yy}
\end{aligned}$$

with the arguments of f_x, f_y, f_{xx}, f_{xy} and f_{yy} all being $\big(r\cos\theta\,,\,r\sin\theta\big)$.

(b) By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial\theta}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= -r\sin\theta\, g_x\big(r\cos\theta\,,\,r\sin\theta\big) + r\cos\theta\, g_y\big(r\cos\theta\,,\,r\sin\theta\big)\\
&= -r\sin\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big) - r\cos\theta\, f_x\big(r\cos\theta\,,\,r\sin\theta\big)\\
&= -r\frac{\partial}{\partial r}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big]\\
\frac{\partial}{\partial r}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= \cos\theta\, g_x\big(r\cos\theta\,,\,r\sin\theta\big) + \sin\theta\, g_y\big(r\cos\theta\,,\,r\sin\theta\big)\\
&= \cos\theta\, f_y\big(r\cos\theta\,,\,r\sin\theta\big) - \sin\theta\, f_x\big(r\cos\theta\,,\,r\sin\theta\big)\\
&= \frac{1}{r}\frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big]
\end{aligned}$$

or

$$\begin{aligned}
\frac{\partial}{\partial r}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= -\frac{1}{r}\frac{\partial}{\partial\theta}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big]\\
\frac{\partial}{\partial\theta}\big[f\big(r\cos\theta\,,\,r\sin\theta\big)\big] &= r\frac{\partial}{\partial r}\big[g\big(r\cos\theta\,,\,r\sin\theta\big)\big]
\end{aligned}$$

5. The temperature in the plane is given by $T(x,y) = e^y\big(x^2+y^2\big)$.

(a) (i) Give the system of equations that must be solved in order to find the warmest and coolest point on the circle $x^2+y^2 = 100$ by the method of Lagrange multipliers.

(ii) Find the warmest and coolest points on the circle by solving that system.

(b) (i) Give the system of equations that must be solved in order to find the critical points of $T(x,y)$.

(ii) Find the critical points by solving that system.

(c) Find the coolest point on the solid disc $x^2+y^2\le 100$.

Solution. By way of preparation, we have

$$\frac{\partial T}{\partial x}(x,y)=2x\,e^y \qquad \frac{\partial T}{\partial y}(x,y)=e^y\big(x^2+y^2+2y\big)$$

(a) (i) For this problem the objective function is $T(x,y)=e^y\big(x^2+y^2\big)$ and the constraint function is $g(x,y)=x^2+y^2-100$. According to the method of Lagrange multipliers, Theorem 2.10.2 in the CLP–III text, we need to find all solutions to

$$T_x=2x\,e^y=\lambda(2x)=\lambda g_x \tag{E1}$$
$$T_y=e^y\big(x^2+y^2+2y\big)=\lambda(2y)=\lambda g_y \tag{E2}$$
$$x^2+y^2=100 \tag{E3}$$

(a) (ii) According to equation (E1), $2x(e^y-\lambda)=0$. This condition is satisfied if and only if at least one of $x=0$, $\lambda=e^y$ is obeyed.

- If $x=0$, then equation (E3) reduces to $y^2=100$, which is obeyed if $y=\pm10$. Equation (E2) then gives the corresponding values for λ, which we don't need.
- If $\lambda=e^y$, then equation (E2) reduces to

  $$e^y\big(x^2+y^2+2y\big)=(2y)e^y \iff e^y\big(x^2+y^2\big)=0$$

  which conflicts with (E3). So we can't have $\lambda=e^y$.

So the only possible locations of the maximum and minimum of the function T are $(0,10)$ and $(0,-10)$. To complete the problem, we only have to compute T at those points.

point	$(0,10)$	$(0,-10)$
value of T	$100e^{10}$	$100e^{-10}$
	max	min

Hence the maximum value of $T(x,y)=e^y\big(x^2+y^2\big)$ on $x^2+y^2=100$ is $100e^{10}$ at $(0,10)$ and the minimum value is $100e^{-10}$ at $(0,-10)$.

We remark that, on $x^2+y^2=100$, the objective function $T(x,y)=e^y\big(x^2+y^2\big)=100e^y$. So of course the maximum value of T is achieved when y is a maximum, i.e. when $y=10$, and the minimum value of T is achieved when y is a minimum, i.e. when $y=-10$.

(b) (i) By definition, the point (x, y) is a critical point of $T(x, y)$ if ane only if

$$T_x = 2x\, e^y = 0 \tag{E1}$$
$$T_y = e^y\big(x^2 + y^2 + 2y\big) = 0 \tag{E2}$$

(b) (ii) Equation (E1) forces $x = 0$. When $x = 0$, equation (E2) reduces to

$$e^y\big(y^2 + 2y\big) = 0 \iff y(y+2) = 0 \iff y = 0 \text{ or } y = -2$$

So there are two critical points, namely $(0, 0)$ and $(0, -2)$.

(c) Note that $T(x, y) = e^y\big(x^2 + y^2\big) \geq 0$ on all of $\mathbb{R}^2$. As $T(x, y) = 0$ only at $(0, 0)$, it is obvious that $(0, 0)$ is the coolest point.

In case you didn't notice that, here is a more conventional solution.

The coolest point on the solid disc $x^2 + y^2 \leq 100$ must either be on the boundary, $x^2 + y^2 = 100$, of the disc or be in the interior, $x^2 + y^2 < 100$, of the disc.

In part (a) (ii) we found that the coolest point on the boundary is $(0, -10)$, where $T = 100e^{-10}$.

If the coolest point is in the interior, it must be a critical point and so must be either $(0, 0)$, where $T = 0$, or $(0, -2)$, where $T = 4e^{-2}$.

So the coolest point is $(0, 0)$.

6. Let $I = \int_0^1 \int_{x^2}^1 x^3 \ \sin(y^3)\ \mathrm{d}y\ \mathrm{d}x$.

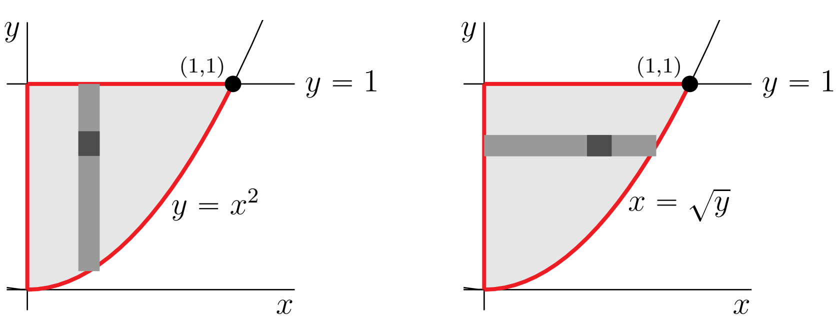
   (a) Sketch the region of integration in the xy–plane. Label your sketch sufficiently well that one could use it to determine the limits of double integration.

   (b) Evaluate I.

Solution. (a) On the domain of integration,

- x runs from 0 to 1 and
- for each fixed x in that range, y runs from x^2 to 1.

The figure on the left below is a sketch of that domain, together with a generic vertical strip as was used in setting up the integral.



(b) As it stands, the inside integral, over y, looks pretty nasty because $\sin(y^3)$ does not have an obvious antiderivative. So let's reverse the order of integration. The given integral was set up using vertical strips. So, to reverse the order of integration, we use horizontal strips as in the figure on the right above. Looking at that figure we see that, on the domain of integration,

- y runs from 0 to 1 and
- for each fixed y in that range, x runs from 0 to $\sqrt{y}$.

So

$$\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_0^{\sqrt{y}} \mathrm{d}x\ x^3\ \sin(y^3) \\
&= \int_0^1 \mathrm{d}y\ \sin(y^3) \left[\frac{x^4}{4}\right]_0^{\sqrt{y}} \\
&= \frac{1}{4}\int_0^1 \mathrm{d}y\ y^2 \sin(y^3) \\
&= \frac{1}{4}\left[-\frac{\cos(y^3)}{3}\right]_0^1 \\
&= \frac{1-\cos(1)}{12}
\end{aligned}$$

7. Let S be the region on the first octant (so that $x, y, z \ge 0$) which lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $(z-1)^2 + x^2 + y^2 = 1$. Let V be its volume.

   (a) Express V as a triple integral in cylindrical coordinates.

   (b) Express V as an triple integral in spherical coordinates.

   (c) Calculate V using either of the integrals above.

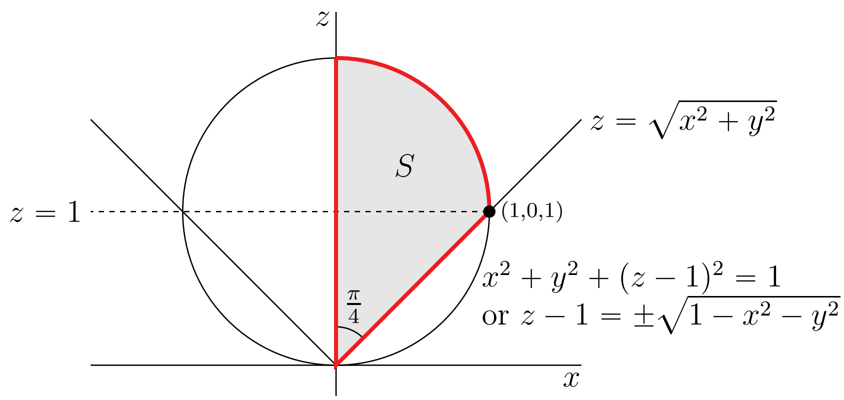
Solution. Note that both the sphere $x^2 + y^2 + (z-1)^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$ are invariant under rotations around the z–axis. The sphere $x^2 + y^2 + (z-1)^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$ intersect when $z = \sqrt{x^2+y^2}$, so that $x^2 + y^2 = z^2$, and

$$\begin{aligned}
x^2 + y^2 + (z-1)^2 = z^2 + (z-1)^2 = 1 &\iff 2z^2 - 2z = 0 \iff 2z(z-1) = 0 \\
&\iff z = 0, 1
\end{aligned}$$

So the surfaces intersect on the circle $z = 1$, $x^2 + y^2 = 1$ and

$$S = \left\{\ (x, y, z)\ \middle|\ x, y \ge 0,\ x^2 + y^2 \le 1,\ \sqrt{x^2+y^2} \le z \le 1 + \sqrt{1 - x^2 - y^2}\ \right\}$$

Here is a sketch of the $y = 0$ cross section of S.



(a) In cylindrical coordinates

- the condition $x, y \geq 0$ is $0 \leq \theta \leq {}^{\pi}/_{2}$,
- the condition $x^2 + y^2 \leq 1$ is $r \leq 1$, and
- the conditions $\sqrt{x^2+y^2} \leq z \leq 1 + \sqrt{1-x^2-y^2}$ are $r \leq z \leq 1 + \sqrt{1-r^2}$, and
- $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$.

So

$$V = \iiint_S \mathrm{d}V = \int_0^1 \mathrm{d}r \int_0^{\pi/2} \mathrm{d}\theta \int_r^{1+\sqrt{1-r^2}} \mathrm{d}z\ r$$

(b) In spherical coordinates,

- the cone $z = \sqrt{x^2+y^2}$ becomes

$$\rho\,\cos\varphi = \sqrt{\rho^2\sin^2\varphi\,\cos^2\theta + \rho^2\sin^2\varphi\,\sin^2\theta} = \rho\sin\varphi \iff \tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$$

- so that, on S, the spherical coordinate φ runs from $\varphi = 0$ (the positive z –axis) to $\varphi = {}^{\pi}/_{4}$ (the cone $z = \sqrt{x^2+y^2}$), which keeps us above the cone,
- the condition $x, y \geq 0$ is $0 \leq \theta \leq {}^{\pi}/_{2}$,
- the condition $x^2 + y^2 + (z-1)^2 \leq 1$, (which keeps us inside the sphere), becomes

$$\begin{aligned}
&\rho^2\sin^2\varphi\,\cos^2\theta + \rho^2\sin^2\varphi\,\sin^2\theta + \big(\rho\cos\varphi - 1\big)^2 \leq 1 \\
&\iff \rho^2\,\sin^2\varphi + \rho^2\cos^2\varphi - 2\rho\cos\varphi + 1 \leq 1 \\
&\iff \rho^2 - 2\rho\cos\varphi \leq 0 \\
&\iff \rho \leq 2\cos\varphi
\end{aligned}$$

- and $\mathrm{d}V = \rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$.

So

$$V = \iiint_S \mathrm{d}V = \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi$$

(c) We'll evaluate V using the spherical coordinate integral of part (b).

$$\begin{aligned} V &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\cos\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi \\ &= \frac{8}{3} \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta\ \cos^3\varphi\ \sin\varphi \\ &= \frac{8}{3}\,\frac{\pi}{2} \left[-\frac{\cos^4\varphi}{4}\right]_0^{\pi/4} \\ &= \frac{\pi}{3}\left[1 - \frac{1}{(\sqrt{2})^4}\right] \\ &= \frac{\pi}{4} \end{aligned}$$

8. Let E be the region bounded by the planes $y = 0$, $y = 2$, $y + z = 3$ and the surface $z = x^2$. Consider the intergal

$$I = \iiint_E f(x, y, z)\ \mathrm{d}V$$

Fill in the blanks below. In each part below, you may need only one integral to express your answer. In that case, leave the other blank.

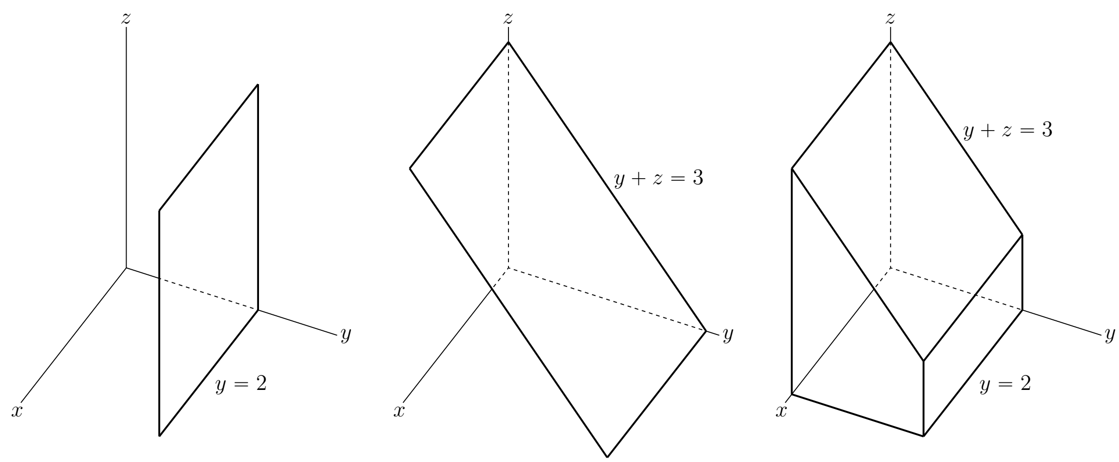
(a) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$

(b) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$

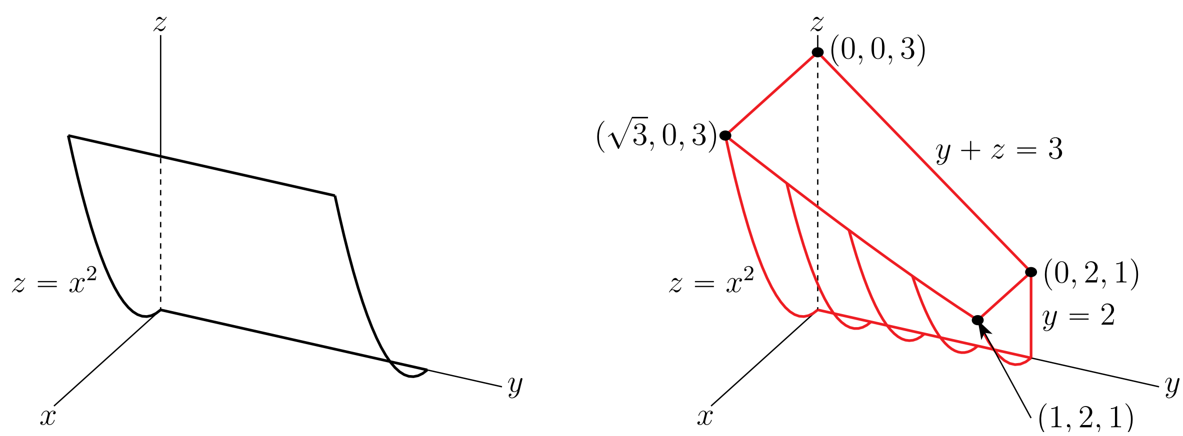
(c) $I = \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z + \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} \int_{\_\_}^{\_\_} f(x, y, z)\ \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}z$

Solution. First, we need to develop an understanding of what E looks like. Note that all of the equations $y = 0$, $y = 2$, $y + z = 3$ and $z = x^2$ are invariant under $x \to -x$. So E is invariant under $x \to -x$, i.e. is symmetric about the yz–plane. We'll sketch the first octant (i.e. $x, y, z \geq 0$) part of E. There is also a $x \leq 0$, $y \geq 0$, $z \geq 0$ part.

Here are sketches of the plane $y = 2$, on the left, the plane $y + z = 3$ in the centre and of the "tunnel" bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and the planes $y = 2$, $y + z = 3$, on the right.



Now here is the parabolic cylinder $z = x^2$ on the left. E is constructed by using the parabolic cylinder $z = x^2$ to chop the front off of the tunnel $x \geq 0$, $0 \leq y \leq 2$, $z \geq 0$, $y + z \leq 3$. The figure on the right is a sketch.



So

$$E = \left\{ \ (x, y, z) \ \middle|\ 0 \leq y \leq 2,\ x^2 \leq z \leq 3 - y \ \right\}$$

(a) On E

- y runs from 0 to 2.
- For each fixed y in that range, (x, z) runs over $\left\{ \ (x, z) \ \middle|\ x^2 \leq z \leq 3 - y \ \right\}$.
- In particular, the largest x^2 is $3 - y$ (when $z = 3 - y$). So x runs from $-\sqrt{3-y}$ to $\sqrt{3-y}$.
- For fixed y and x as above, z runs from x^2 to $3 - y$.

so that

$$I = \iiint_E f(x, y, z)\ \mathrm{d}V = \int_0^2 \int_{-\sqrt{3-y}}^{\sqrt{3-y}} \int_{x^2}^{3-y} f(x, y, z)\ \mathrm{d}z\, \mathrm{d}x\, \mathrm{d}y$$

(b) On E

- z runs from 0 to 3.
- For each fixed z in that range, (x, y) runs over

  $$\left\{ (x,y) \;\middle|\; 0 \le y \le 2,\ x^2 \le z \le 3-y \right\} = \left\{ (x,y) \;\middle|\; 0 \le y \le 2,\ y \le 3-z,\ x^2 \le z \right\}$$

  In particular, y runs from 0 to the minimum of 2 and $3 - z$.
- So if $0 \le z \le 1$ (so that $3 - z \ge 2$), (x, y) runs over $\left\{ (x,y) \;\middle|\; 0 \le y \le 2,\ x^2 \le z \right\}$, while
- if $1 \le z \le 3$, (so that $3 - z \le 2$), (x, y) runs over $\left\{ (x,y) \;\middle|\; 0 \le y \le 3-z,\ x^2 \le z \right\}$,

so that

$$I = \int_0^1 \int_0^2 \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_1^3 \int_0^{3-z} \int_{-\sqrt{z}}^{\sqrt{z}} f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

(c) On E

- z runs from 0 to 3.
- For each fixed z in that range, (x, y) runs over

  $$\left\{ (x,y) \;\middle|\; 0 \le y \le 2,\ x^2 \le z \le 3-y \right\}$$

  In particular, y runs from 0 to the minimum of 2 and $3 - z$.
- So if $0 \le z \le 1$ (so that $3 - z \ge 2$), (x, y) runs over $\left\{ (x,y) \;\middle|\; 0 \le y \le 2,\ x^2 \le z \right\}$, while
- if $1 \le z \le 3$, (so that $3 - z \le 2$), (x, y) runs over $\left\{ (x,y) \;\middle|\; 0 \le y \le 3-z,\ x^2 \le z \right\}$,

so that

$$I = \int_0^1 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^2 f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z + \int_1^3 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^{3-z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$$