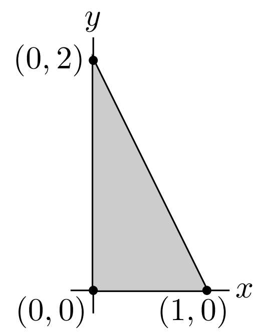
Global Maximum/Minimum Example



Problem. Find the maximum and minimum values of

$$f(x, y) = xy + 2x + y$$

on the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$.

Solution.

Interior. Since

$$f_x(x, y) = y + 2 \qquad f_y(x, y) = x + 1$$

there are no singular points and the only critical point, $(-1, -2)$, is not in the triangular region.

Side $x = 0$, $0 \le y \le 2$. On that side $f(0, y) = y$ and

$$\min_{0\le y\le 2} f(0, y) = f(0, 0) = 0 \qquad \max_{0\le y\le 2} f(0, y) = f(0, 2) = 2$$

Base $y = 0$, $0 \le x \le 1$. On that side $f(x, 0) = 2x$ and

$$\min_{0\le x\le 1} f(x, 0) = f(0, 0) = 0 \qquad \max_{0\le x\le 1} f(x, 0) = f(1, 0) = 2$$

Hypotenuse. On that side $y = 2 - 2x$, $0 \le x \le 1$ and

$$f(x, \overbrace{2 - 2x}^{y}) = x(\overbrace{2 - 2x}^{y}) + 2x + (\overbrace{2 - 2x}^{y}) = -2x^2 + 2x + 2$$

Write $g(x) = -2x^2 + 2x + 2$. The maximum and minimum of $g(x)$ for $0 \le x \le 1$, and hence the maximum and minimum values of f on the hypotenuse of the triangle, must be achieved either at

- $x = 0$, where $f(0, 2) = g(0) = 2$, or at
- $x = 1$, where $f(1, 0) = g(1) = 2$, or when
- $0 = g'(x) = -4x + 2$ so that $x = \frac{1}{2}$, $y = 2 - 2\left(\frac{1}{2}\right) = 1$ and

$$f\left(\tfrac{1}{2}, 1\right) = g\left(\tfrac{1}{2}\right) = -2\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right) + 2 = \tfrac{5}{2}$$

Candidates. Here are all the candidates for the location of a max or min.

point	$(0, 0)$	$(0, 2)$	$(1, 0)$	$\left(\frac{1}{2}, 1\right)$
value of f	0	2	2	$\frac{5}{2}$
	min			max

Finding the Equation of the Line Through $(0,2)$ and $(1,0)$

Method 1: using $Ax + By = 1$.

- Every line in the xy-plane has an equation of the form $ax + by = c$.
- In this case $(0,0)$ is **not** on the line so that $c \neq 0$ and we can divide the equation by c, giving $\frac{a}{c}x + \frac{b}{c}y = 1$. Rename $\frac{a}{c} = A$ and $\frac{b}{c} = B$.
- $(0,2)$ is on the line so that

$$Ax\Big|_{x=0} + By\Big|_{y=2} = 1 \implies B = \tfrac{1}{2}$$

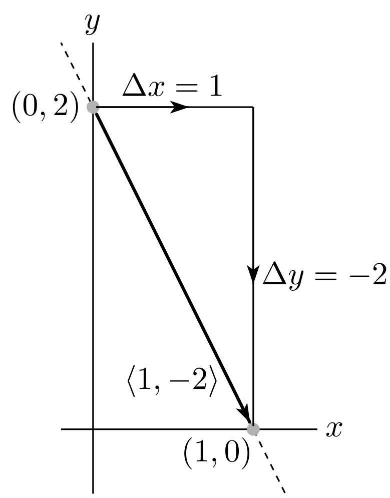
- $(1,0)$ is on the line so that

$$Ax\Big|_{x=1} + By\Big|_{y=0} = 1 \implies A = 1$$

- So the line is $x + \frac{y}{2} = 1$ or $y = 2 - 2x$.

Method 2: using $y = mx + b$.

- b is the y-intercept, i.e. the y-coordinate of the point on the line where $x = 0$. In this case $b = 2$.
- m is the slope. In this case $m = \frac{\Delta y}{\Delta x} = \frac{0-2}{1-0} = -2$.
- So the line is $y = 2 - 2x$.



Method 2: using parameterization.

- $(0,2)$ is one point on the line.
- the vector from $(0,2)$ to $(1,0)$, namely $\langle 1-0\,,\,0-2\rangle = \langle 1,-2\rangle$, is a direction vector for the line.
- So the line is $\langle x-0\,,\,y-2\rangle = t\,\langle 1,-2\rangle$ or $x = t$, $y = 2 - 2t$.