Taylor Expansions in 2d

In your first year Calculus course you developed a family of formulae for approximating a function $F(t)$ for t near any fixed point t_0.

$$\begin{aligned}
F(t_0+\Delta t) &\approx F(t_0) \\
F(t_0+\Delta t) &\approx F(t_0) + F'(t_0)\Delta t \\
F(t_0+\Delta t) &\approx F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2}F''(t_0)\Delta t^2 \\
&\ \ \vdots \\
F(t_0+\Delta t) &\approx F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2!}F''(t_0)\Delta t^2 \\
&\qquad + \tfrac{1}{3!}F^{(3)}(t_0)\Delta t^3 + \cdots + \tfrac{1}{n!}F^{(n)}(t_0)\Delta t^n
\end{aligned}$$

You may have also found a formula for the error introduced in making this approximation. The error $E_n(\Delta t)$ is defined by

$$\begin{aligned}
F(t_0+\Delta t) = F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2!}F''(t_0)\Delta t^2 \\
+ \cdots + \tfrac{1}{n!}F^{(n)}(t_0)\Delta t^n + E_n(\Delta t)
\end{aligned}$$

and obeys

$$E_n(\Delta t) = \tfrac{1}{(n+1)!}F^{(n+1)}(t^*)\Delta t^{n+1}$$

for some (unknown) t^* between t_0 and $t_0 + \Delta t$.

We now generalize to functions of more than one variable. Suppose we wish to approximate $f(x_0 + \Delta x, y_0 + \Delta y)$ for Δx and Δy near zero. The trick is to write

$$f(x_0+\Delta x, y_0+\Delta y) = F(1) \text{ with } F(t) = f(x_0+t\Delta x, y_0+t\Delta y)$$

and think of x_0, y_0, Δx and Δy as constants so that F is a function of the single variable t. Then we can apply our single variable formulae with $t_0 = 0$ and $\Delta t = 1$. To do so we need to compute various derivatives of $F(t)$ at $t = 0$, by applying the chain rule to

$$F(t) = f\big(x(t), y(t)\big) \text{ with } x(t) = x_0 + t\Delta x,\ \ y(t) = y_0 + t\Delta y$$

Since $\frac{d}{dt}x(t) = \Delta x$ and $\frac{d}{dt}y(t) = \Delta y$, the chain rule gives

$$\begin{aligned}
\tfrac{dF}{dt}(t) &= \tfrac{\partial f}{\partial x}\big(x(t), y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f}{\partial y}\big(x(t), y(t)\big)\tfrac{d}{dt}y(t) \\
&= f_x\big(x(t), y(t)\big)\Delta x + f_y\big(x(t), y(t)\big)\Delta y \\
\tfrac{d^2F}{dt^2}(t) &= \Big[\tfrac{\partial f_x}{\partial x}\big(x(t), y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f_x}{\partial y}\big(x(t), y(t)\big)\tfrac{d}{dt}y(t)\Big]\Delta x \\
&\quad + \Big[\tfrac{\partial f_y}{\partial x}\big(x(t), y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f_y}{\partial y}\big(x(t), y(t)\big)\tfrac{d}{dt}y(t)\Big]\Delta y \\
&= f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2
\end{aligned}$$

and so on. It's not hard to prove by induction that, in general,

$$F^{(n)}(t) = \sum_{m=0}^{n} \binom{n}{m} \tfrac{\partial^n f}{\partial x^m \partial y^{n-m}}(x_0+t\Delta x, y_0+t\Delta y)\Delta x^m \Delta y^{n-m}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ is the standard binomial coefficient. So when $t = 0$,

$$\begin{aligned} F(0) &= f_{(x_0,y_0)} \\ \frac{dF}{dt}(0) &= f_{x(x_0,y_0)}\Delta x + f_{y(x_0,y_0)}\Delta y \\ \frac{d^2F}{dt^2}(0) &= f_{xx(x_0,y_0)}\Delta x^2 + 2f_{xy(x_0,y_0)}\Delta x\Delta y + f_{yy(x_0,y_0)}\Delta y^2 \end{aligned}$$

Subbing these into

$$\begin{aligned} f(x_0+\Delta x, y_0+\Delta y) &= F(t_0+\Delta t)\Big|_{t_0=0,\Delta t=1} \\ &= F(0) + F'(0) + \tfrac{1}{2}F''(0) + \cdots \end{aligned}$$

gives

$$\boxed{\begin{aligned} &f(x_0+\Delta x, y_0+\Delta y) \\ &\quad = f_{(x_0,y_0)} \\ &\quad + f_{x(x_0,y_0)}\Delta x + f_{y(x_0,y_0)}\Delta y \\ &\quad + \tfrac{1}{2}\Big[f_{xx(x_0,y_0)}\Delta x^2 + 2f_{xy(x_0,y_0)}\Delta x\Delta y + f_{yy(x_0,y_0)}\Delta y^2\Big] \\ &\quad + O\big(\Delta x^3 + \Delta y^3\big) \end{aligned}}$$