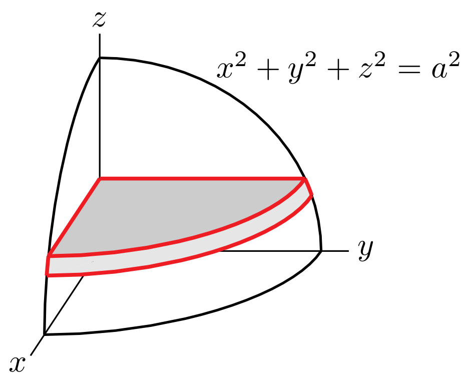
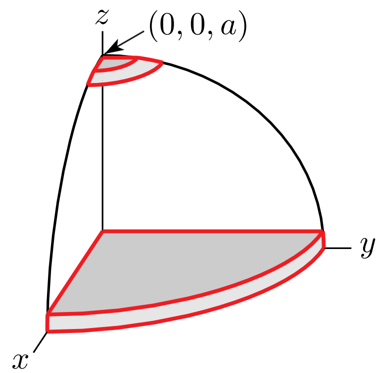
Triple Integral Example

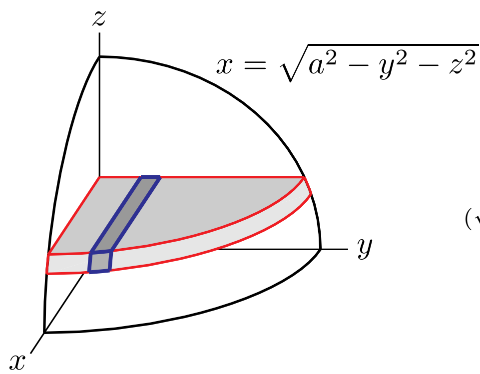
In this example, we find the mass of the part of the first octant that is inside the sphere $x^2 + y^2 + z^2 = a^2$, if the density is $\rho(x, y, z) = xy$.

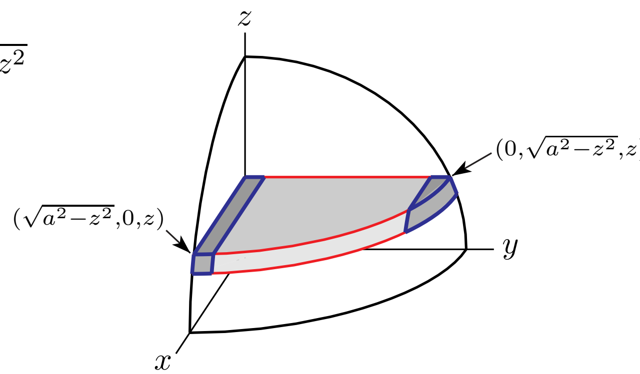
- Slice the solid into horizontal plates by inserting many planes of constant z, with the various values of z differing by $\mathrm{d}z$.





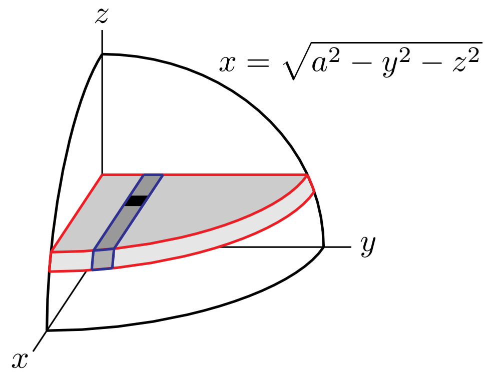
  - Each plate has thickness $\mathrm{d}z$, and
  - has z almost constant throughout the plate (it only varies by $\mathrm{d}z$), and
  - has (x, y) running over $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq a^2 - z^2$.
  - The bottom plate starts at $z = 0$ and the top plate ends at $z = a$.
- Concentrate on any one plate. Subdivide it into long thin "square" beams by inserting many planes of constant y, with the various values of y differing by $\mathrm{d}y$.

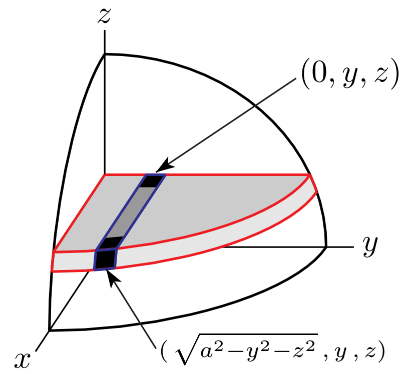




  - Each beam has cross-sectional area $\mathrm{d}y\,\mathrm{d}z$, and
  - has y and z essentially constant throughout the beam, and
  - has x running over $0 \leq x \leq \sqrt{a^2 - y^2 - z^2}$.
  - The leftmost beam has $y = 0$ and the rightmost beam has $y = \sqrt{a^2 - z^2}$.

- Concentrate on any one beam. Subdivide it into tiny approximate "cubes" by inserting many planes of constant x, with the various values of x differing by $\mathrm{d}x$.





  - Each cube has volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, and
  - has x, y and z all essentially constant throughout the cube.
  - The first cube has $x = 0$ and the last cube has $x = \sqrt{a^2 - y^2 - z^2}$.
- Now we can build up the mass.
  - Concentrate on one approximate cube, say the cube containing the point (x, y, z).
    - That cube has volume essentially $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and
    - essentially has density $\rho(x, y, z) = xy$ and so
    - essentially has mass $\rho(x, y, z)\,\mathrm{d}V = xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.
  - To get the mass of any one beam, say the beam in the figure above, we just add up the masses of the approximate cubes in that beam, by integrating x from its smallest value on the beam, namely 0, to its largest value on the beam, namely $\sqrt{a^2 - y^2 - z^2}$. The mass of the beam is thus

$$\mathrm{d}y\,\mathrm{d}z \int_0^{\sqrt{a^2-y^2-z^2}} \mathrm{d}x\; xy$$

  - To get the mass of any one plate, say the plate in the figure above, we just add up the masses of the beams in that plate, by integrating y from its smallest value on the plate, namely 0, to its largest value on the plate, namely $\sqrt{a^2 - z^2}$. The mass of the plate is thus

$$\mathrm{d}z \int_0^{\sqrt{a^2-z^2}} \mathrm{d}y \int_0^{\sqrt{a^2-y^2-z^2}} \mathrm{d}x\; xy$$

  - To get the mass of the whole solid, we just add up the masses of the plates that it contains, by integrating z from its smallest value, namely 0, to its largest value on the solid, namely a.

The mass of the solid is thus

$$
\begin{aligned}
\int_0^a \mathrm{d}z \int_0^{\sqrt{a^2-z^2}} \mathrm{d}y \int_0^{\sqrt{a^2-y^2-z^2}} \mathrm{d}x\, xy &= \int_0^a \mathrm{d}z \int_0^{\sqrt{a^2-z^2}} \mathrm{d}y\ y \left[\int_0^{\sqrt{a^2-y^2-z^2}} x\ \mathrm{d}x\right] \\
&= \int_0^a \mathrm{d}z \int_0^{\sqrt{a^2-z^2}} \mathrm{d}y\ \frac{y}{2}\left(a^2-y^2-z^2\right) \\
&= \int_0^a \mathrm{d}z \left[\int_0^{\sqrt{a^2-z^2}} \left\{\frac{(a^2-z^2)}{2}y - \frac{y^3}{2}\right\}\ \mathrm{d}y\right] \\
&= \int_0^a \mathrm{d}z\ \overbrace{\left\{\frac{\left(a^2-z^2\right)^2}{4} - \frac{\left(a^2-z^2\right)^2}{8}\right\}}^{\frac{1}{8}(a^2-z^2)^2} \\
&= \frac{1}{8}\int_0^a \left[a^4 - 2a^2z^2 + z^4\right]\ \mathrm{d}z \\
&= \frac{1}{8}\left[a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right] \\
&= \frac{a^5}{8}\ \frac{15-10+3}{15} \\
&= \boxed{\frac{a^5}{15}}
\end{aligned}
$$