# Approximating Functions Near a Specified Point

Suppose that you are interested in the values of some function $f(x)$ for $x$ near some fixed point $x_0$. The function is too complicated to work with directly. So you wish to work instead with some other function $F(x)$ that is both simple and a good approximation to $f(x)$ for $x$ near $x_0$. We'll consider an example of this scenario later. First, we use the Fundamental Theorem of Calculus and integration by parts to develop several different approximations.

The Fundamental Theorem of Calculus says that $\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$, or equivalently,

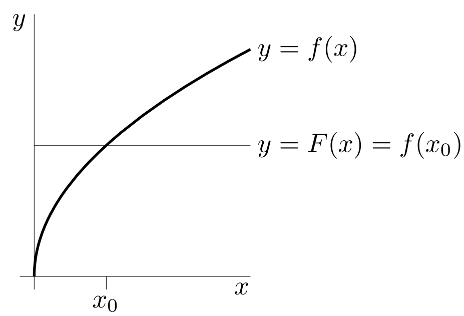
$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt \tag{1'}$$

### First approximation

The simplest functions are those that are constants. Our first, and crudest, approximation is the constant function $f(x_0)$.

$$\boxed{f(x) \approx f(x_0)} \tag{1}$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. At $x = x_0$, $f(x)$ and




$F(x)$ take the same value. For $x$ very near $x_0$, the values of $f(x)$ and $F(x)$ remain close together. But the quality of the approximation deteriorates fairly quickly as $x$ moves away from $x_0$. Equation (1′) says that the error you make when you approximate $f(x)$ by $f(x_0)$ (namely $|fx) - f(x_0)|$) is exactly $\left|\int_{x_0}^{x} f'(t)\,dt\right|$. Usually this integral is too complicated to evaluate in a useful way (or you wouldn't be approximating $f(x)$ in this first place). But if you can find a constant $M_1$ such that $|f'(t)| \le M_1$ for all $t$ between $x_0$ and $x$ (we'll see examples of this later), then

$$\text{error} = \left|\int_{x_0}^{x} f'(t)\,dt\right| \le \left|\int_{x_0}^{x} M_1\,dt\right| = M_1|x - x_0|$$

So the error grows at most linearly as $x$ moves away from $x_0$.

### Second approximation – the tangent line, or linear, approximation

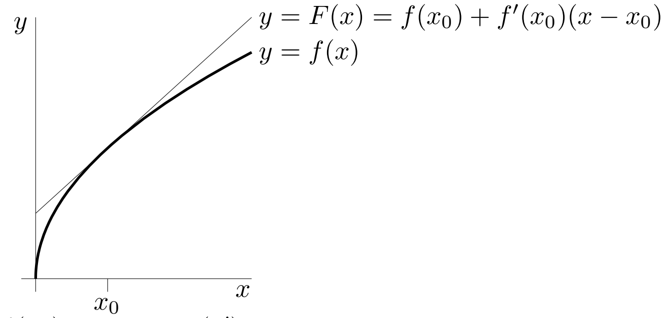
We now develop a better approximation by applying integration by parts to the integral in (1′). Note that the integration variable is $t$. As far as the integration is concerned, $x$ is a constant. We use $u(t) = f'(t)$ and $v(t) = x - t$. Since $x$ is a constant, $dv = v'(t)dt = -dt$.

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(t)\,dt = f(x_0) - \int_{x_0}^{x} u(t)v'(t)\,dt = f(x_0) - u(x)v(x) + u(x_0)v(x_0) + \int_{x_0}^{x} v(t)u'(t)\,dt \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} (x - t)f''(t)\,dt \end{aligned} \tag{2'}$$

Our second approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)} \tag{2}$$

It is exactly the tangent line approximation to $f(x)$ at $x_0$. The original function $f(x)$ and the approximating function $F(x) = f(x_0)+f'(x_0)(x-x_0)$ have the same value and the same slope at $x = x_0$. That is, $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$. Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. Observe that the graph of $f(x_0) + f'(x_0)(x - x_0)$ remains close to the graph of $f(x)$ for a much larger range




of $x$ than did the graph of $f(x_0)$. Equation (2′) says that the error you make when you approximate $f(x)$ by $f(x_0) + f'(x_0)(x - x_0)$ is exactly $\left|\int_{x_0}^{x}(x-t)f''(t)\,dt\right|$. Again, this integral is usually too complicated to evaluate in a useful way. But if you can find a constant $M_2$ such that $|f''(t)| \le M_2$ for all $t$ between $x_0$ and $x$, then

$$\text{error} = \left|\int_{x_0}^{x}(x-t)f''(t)\,dt\right| \le \left|\int_{x_0}^{x}(x-t)M_2\,dt\right| = \left|-M_2\tfrac{(x-t)^2}{2}\Big|_{t=x_0}^{t=x}\right| = \tfrac{M_2}{2}(x-x_0)^2$$

So the error grows at most quadratically as $x$ moves away from $x_0$.

**Third approximation – the quadratic approximation**

We develop a still better approximation by applying integration by parts a second time – this time to the integral in (2′). We use $u(t) = f''(t)$ and $v(t) = \frac{1}{2}(x - t)^2$, so that $v'(t) = -(x - t)$.

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x}(x-t)f''(t)\,dt = f(x_0) + f'(x_0)(x-x_0) - \int_{x_0}^{x}u(t)v'(t)\,dt \\
&= f(x_0) + f'(x_0)(x-x_0) - u(x)v(x) + u(x_0)v(x_0) + \int_{x_0}^{x}v(t)u'(t)\,dt \\
&= f(x_0) + f'(x_0)(x-x_0) - \tfrac{1}{2}f''(x)(x-x)^2 + \tfrac{1}{2}f''(x_0)(x-x_0)^2 + \tfrac{1}{2}\int_{x_0}^{x}(x-t)^2 f^{(3)}(t)\,dt \\
&= f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2}f''(x_0)(x-x_0)^2 + \tfrac{1}{2}\int_{x_0}^{x}(x-t)^2 f^{(3)}(t)\,dt
\end{aligned} \tag{3′}$$

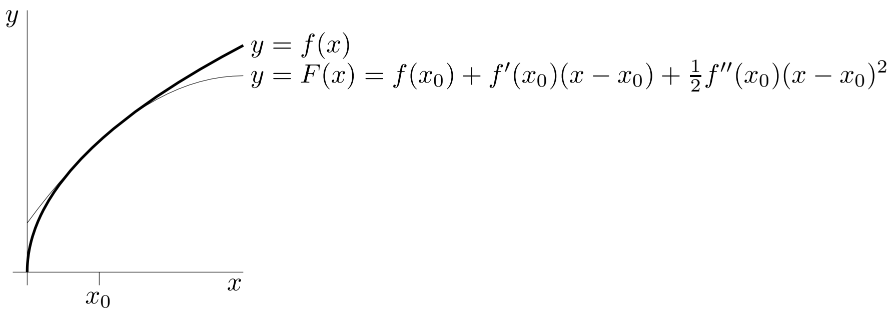
Our third approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2}f''(x_0)(x-x_0)^2} \tag{3}$$

It is called the quadratic approximation. Equation (3′) says that the error you make when you approximate $f(x)$ by $f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$ is exactly $\left|\frac{1}{2}\int_{x_0}^{x}(x-t)^2 f^{(3)}(t)\,dt\right|$. If you can find a constant $M_3$ such that $|f^{(3)}(t)| \le M_3$ for all $t$ between $x_0$ and $x$ , then

$$\text{error} = \left|\tfrac{1}{2}\int_{x_0}^{x}(x-t)^2 f^{(3)}(t)\,dt\right| \le \left|\tfrac{1}{2}\int_{x_0}^{x}(x-t)^2 M_3\,dt\right| = \left|-M_3\tfrac{(x-t)^3}{2\times 3}\Big|_{t=x_0}^{t=x}\right| = \tfrac{M_3}{3!}|x-x_0|^3$$

where 3! (read "three factorial") means $1 \times 2 \times 3$. Here is a figure showing the graphs of a typical $f(x)$ and quadratic approximating function $F(x)$. The third approximation looks better than both the first and second.




**Still better approximations – Taylor polynomials**

We can use the same strategy to generate still better approximations by polynomials of any degree we like. Integrating by parts repeatedly gives, for any natural number $n$,

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2!}f''(x_0)(x-x_0)^2 + \tfrac{1}{3!}f^{(3)}(x_0)(x-x_0)^3 + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)(x-x_0)^n \\ &\quad + \tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt \end{aligned} \tag{4'}$$

If you can find a constant $M_{n+1}$ such that $|f^{(n+1)}(t)| \le M_{n+1}$ for all $t$ between $x_0$ and $x$, then the error in the approximation

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2!}f''(x_0)(x-x_0)^2 + \tfrac{1}{3!}f^{(3)}(x_0)(x-x_0)^3 + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)(x-x_0)^n} \tag{4}$$

is at most

$$\text{error} = \left|\tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt\right| \le \left|\tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n M_{n+1}\,dt\right| = \left| -M_{n+1}\tfrac{(x-t)^{n+1}}{n!\times(n+1)}\Big|_{t=x_0}^{t=x}\right| = \tfrac{M_{n+1}}{(n+1)!}|x-x_0|^{n+1}$$

The right hand side of (4) is called the Taylor polynomial of degree $n$ for $f$.

There is a second formula for the error that we can derive easily from $E_n = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt$. The only assumption that we shall make is that $f^{(n+1)}(t)$ is continuous in $t$ for $t$ running between $x_0$ and $x$. Let $m$ and $M$ denote the largest and smallest values, respectively, that $f^{(n+1)}(t)$ takes for $t$ between $x_0$ and $x$. Then for each $t$ between $x_0$ and $x$, the integrand $(x-t)^n f^{(n+1)}(t)$ must lie between $(x-t)^n m$ and $(x-t)^n M$. So $E_n$ must lie between

$$\tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n m\,dt = \tfrac{m}{(n+1)!}(x-x_0)^{n+1} \text{ and } \tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n M\,dt = \tfrac{M}{(n+1)!}(x-x_0)^{n+1}$$

That is, $\dfrac{E_n}{\frac{1}{(n+1)!}(x-x_0)^{n+1}}$ must lie between $m$ and $M$. Since $f^{(n+1)}(t)$ is continuous and takes the values $m$ and $M$ for some $t$'s between $x_0$ and $x$, $f^{(n+1)}(t)$ must take all values between $m$ and $M$ as $t$ runs from $x_0$ to $x$. In particular there must exist a $t^*$ between $x_0$ and $x$ such that

$$f^{(n+1)}(t^*) = \frac{E_n}{\frac{1}{(n+1)!}(x-x_0)^{n+1}} \;\Rightarrow\; E_n = \tfrac{1}{(n+1)!}f^{(n+1)}(t^*)(x-x_0)^{n+1}$$

In conclusion, we have that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2!}f''(x_0)(x-x_0)^2 + \tfrac{1}{3!}f^{(3)}(x_0)(x-x_0)^3 + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + E_n$$

with

$$E_n = \tfrac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt = \tfrac{1}{(n+1)!}f^{(n+1)}(t^*)(x-x_0)^{n+1}$$

for some $t$ between $x_0$ and $x$.

**Example**

Let's find all Taylor polyomial for $\sin x$ and $\cos x$ at $x_0 = 0$. To do so, we merely need compute all derivatives of $\sin x$ and $\cos x$ at $x_0 = 0$. First, compute all derivatives at general $x$.

$$\begin{array}{llllll} f(x)=\sin x & f'(x)=\cos x & f''(x)=-\sin x & f^{(3)}(x)=-\cos x & f^{(4)}(x)=\sin x & \cdots \\ g(x)=\cos x & g'(x)=-\sin x & g''(x)=-\cos x & g^{(3)}(x)=\sin x & g^{(4)}(x)=\cos x & \cdots \end{array}$$

The pattern starts over again with the fourth derivative being the same as the original function. Now set $x = x_0 = 0$.

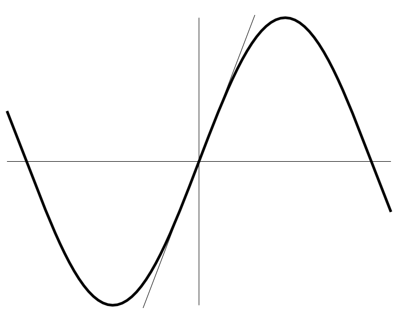
$$\begin{array}{lllllll} f(x)=\sin x & f(0)=0 & f'(0)=1 & f''(0)=0 & f^{(3)}(0)=-1 & f^{(4)}(0)=0 & \cdots \\ g(x)=\cos x & g(0)=1 & g'(0)=0 & g''(0)=-1 & g^{(3)}(0)=0 & g^{(4)}(0)=1 & \cdots \end{array}$$

For $\sin x$, all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and $-1$. For $\cos x$, all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and $-1$. So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = x_0 = 0$ are

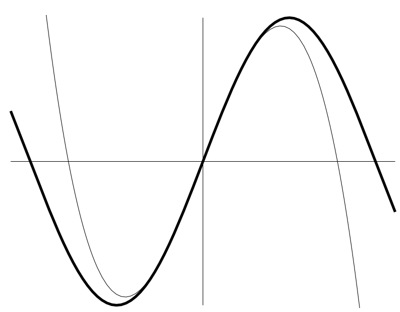
$$\sin x \approx x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots \qquad \cos x \approx 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots$$

Note that all of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians. When applying these approximation formulae, which were developed using trig function derivatives, we are obliged to express $x$ in radians.

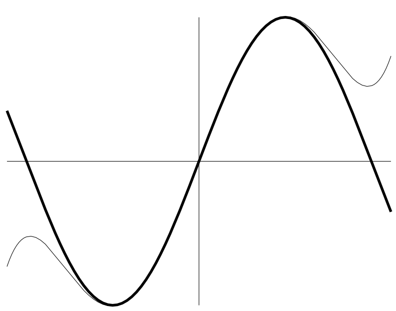
Here are graphs of $\sin x$ and its Taylor poynomials (about $x_0 = 0$) up to degree seven.



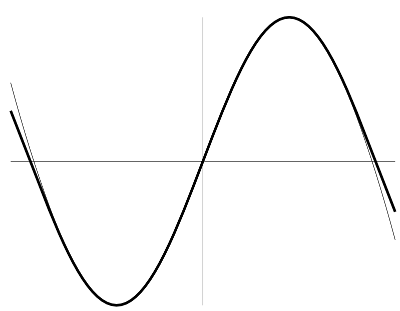
$\sin x \approx x$



$\sin x \approx x - \frac{1}{3!}x^3$



$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$



$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$

Visually, we cannot distinguish between $\sin x$ and the approximation $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$ for $|x|$ running from 0 almost to $\pi$.

Now imagine that we are designing a scientific pocket calculator. To have sin and cos buttons on the calculator we need algorithms for computing $\sin x$ and $\cos x$ accurate to nine decimal places. Because sin and cos have period $2\pi$, it suffices to consider $-\pi \leq x \leq \pi$. Because sin is odd and cos is even, it suffices to consider $0 \leq x \leq \pi$. Because $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, it suffices to consider $0 \leq x \leq \frac{\pi}{2}$. Because $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, it suffices to consider $0 \leq x \leq \frac{\pi}{4}$. Using double angle formulae, it is possible to cut down the range of $x$'s even further, but let's stop here and design algorithms for computing $\sin x$ and $\cos x$ accurate to nine decimal places when $0 \leq x \leq \frac{\pi}{4}$. We have already seen that every derivative of $f(x) = \sin x$ and $g(x) = \cos x$ is either $\sin x$ or $\cos x$ or $-\sin x$ or $-\cos x$. Thus every derivative of $f(x) = \sin x$ and $g(x) = \cos x$ never has magnitude bigger than one. So applying the approximation (4) introduces an error no large than $\frac{M_{n+1}}{(n+1)!}|x - x_0|^{n+1}$ with $M_{n+1} = 1$. When $x_0 = 0$ and $0 \leq x \leq \frac{\pi}{4}$, the error is at most $\frac{1}{(n+1)!}\left(\frac{\pi}{4}\right)^{n+1}$. Here is a table giving values of $\frac{1}{(n+1)!}\left(\frac{\pi}{4}\right)^{n+1}$ (accurate to two significant digits) for various values of $n$.

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|
| $\frac{1}{(n+1)!}\left(\frac{\pi}{4}\right)^{n+1}$ | $3.3 \times 10^{-4}$ | $3.7 \times 10^{-5}$ | $3.6 \times 10^{-6}$ | $3.1 \times 10^{-7}$ | $2.5 \times 10^{-8}$ | $1.8 \times 10^{-9}$ | $1.2 \times 10^{-10}$ |

So applying the approximation (4) with $n = 11$ gives

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} \qquad \cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10}$$

with errors that must be smaller than $1.2 \times 10^{-10}$ when $0 \leq x \leq \frac{\pi}{4}$.