A Fubini Counterexample

Fubini's Theorem states

Theorem (Fubini) *If $f(x,y)$ is continuous in a region R described by both*

$$x_1 \le x \le x_2 \qquad y_1(x) \le y \le y_2(x)$$

and

$$y_1 \le y \le y_2 \qquad x_1(y) \le x \le x_2(y)$$

with $y_1(x)$, $y_2(x)$, $x_1(y)$ and $x_2(y)$ continuous, then

$$\int_{x_1}^{x_2} dx \Big[\int_{y_1(x)}^{y_2(x)} dy\ f(x,y)\Big] \qquad \textit{and} \qquad \int_{y_1}^{y_2} dy \Big[\int_{x_1(y)}^{x_2(y)} dx\ f(x,y)\Big]$$

both exist and are equal.

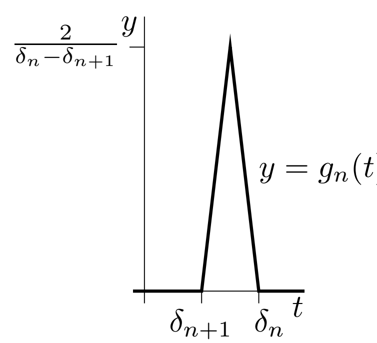
In these notes, we relax exactly one of the hypotheses of Fubini's Theorem, namely the continuity of f, and construct an example in which both of the integrals in Fubini's Theorem exist, but are **not equal**. In fact, we choose $R = \big\{\ (x,y)\ \big|\ 0 \le x \le 1,\ 0 \le y \le 1\ \big\}$ and we use a function $f(x,y)$ that is continuous on R, except at exactly one point – the origin.

First, let $\delta_1, \delta_2, \delta_3, \cdots$ be any sequence of real numbers obeying

$$1 = \delta_1 > \delta_2 > \delta_3,\ \cdots,\ \delta_n \to 0$$

For example $\delta_n = \frac{1}{n}$ or $\delta_n = \frac{1}{2^{n-1}}$ are both acceptable. For each positive integer n, let $I_n = (\delta_{n+1}, \delta_n] = \big\{\ t\ \big|\ \delta_{n+1} < t \le \delta_n\ \big\}$ and let $g_n(t)$ be any continuous function obeying $g_n(\delta_{n+1}) = g_n(\delta_n) = 0$ and $\int_{I_n} g(t)\,dt = 1$. There are many such functions. For example

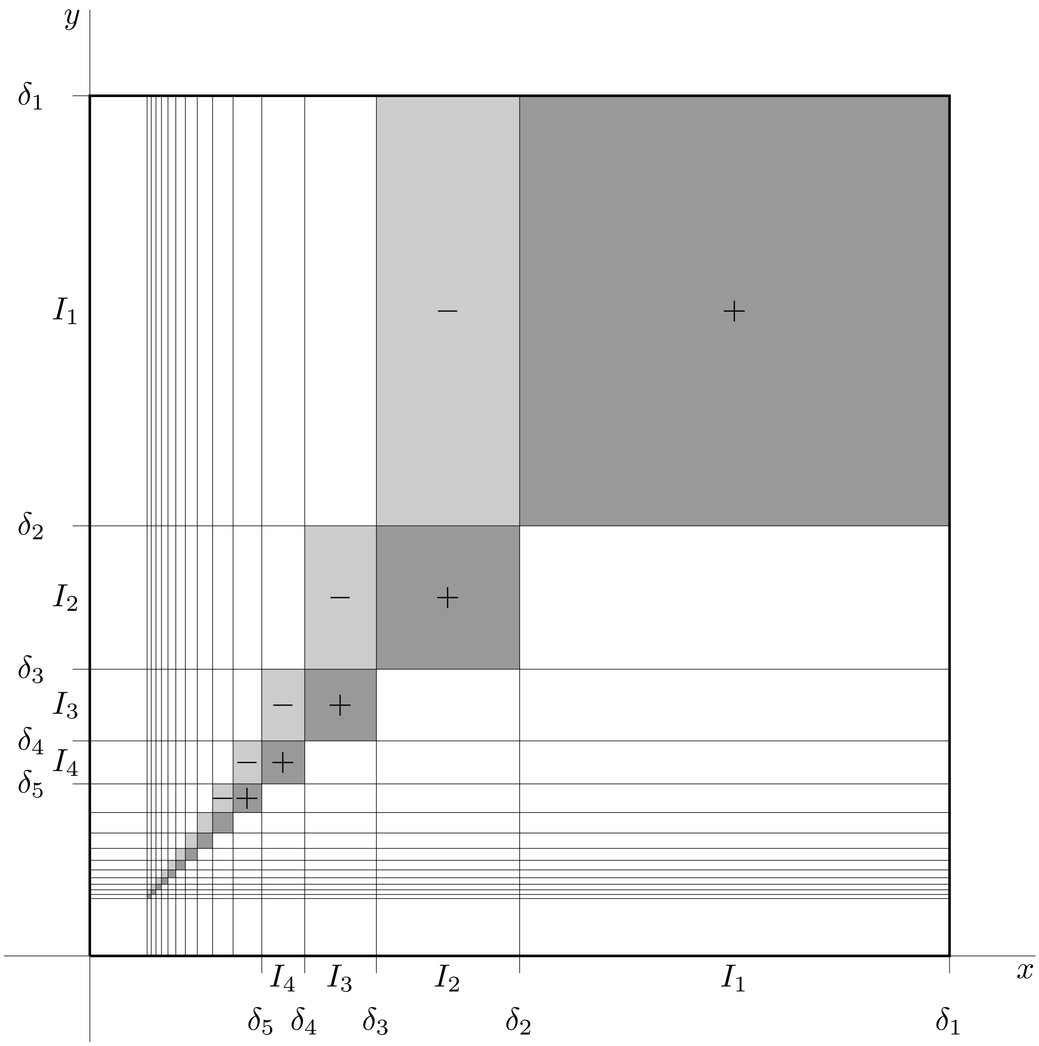
$$g_n(t) = \tfrac{2}{\delta_n - \delta_{n+1}} \begin{cases} \delta_n - t & \text{if } \frac{1}{2}(\delta_{n+1} + \delta_n) \le t \le \delta_n \\ t - \delta_{n+1} & \text{if } \delta_{n+1} \le t \le \frac{1}{2}(\delta_{n+1} + \delta_n) \\ 0 & \text{otherwise} \end{cases}$$



Now define

$$f(x,y) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } y = 0 \\ g_m(x) g_n(y) & \text{if } x \in I_m,\ y \in I_n \text{ with } m = n \\ -g_m(x) g_n(y) & \text{if } x \in I_m,\ y \in I_n \text{ with } m = n+1 \\ 0 & \text{otherwise} \end{cases}$$

You should think of $(0,1] \times (0,1]$ as a union of a bunch of small rectangles $I_m \times I_n$, as in the figure below. On most of these rectangles, $f(x,y)$ is just zero. The exceptions are the darkly shaded rectangles $I_n \times I_n$ on the "diagonal" of the figure and the lightly shaded rectanges $I_{n+1} \times I_n$ just to the left of the "diagonal". On each darkly shaded rectangle, the graph of $f(x,y)$ is the graph of $g_n(x)g_n(y)$ which looks like a pyramid. On each lightly shaded rectangle, the graph of $f(x,y)$ is the graph of $-g_{n+1}(x)g_n(y)$ which looks like a pyramidal hole in the ground.



Now fix any $0 \le y \le 1$ and let's compute $\int_0^1 f(x,y)\,dx$. That is, we are integrating f along a line that is parallel to the x–axis. If $y = 0$, then $f(x,y) = 0$ for all x, so $\int_0^1 f(x,y)\,dx = 0$. If $0 < y \le 1$, then there is exactly one positive integer n with $y \in I_n$ and $f(x,y)$ is zero, except for x in I_n or I_{n+1}. So for $y \in I_n$

$$\begin{aligned}\int_0^1 f(x,y)\,dx &= \sum_{m=n,n+1} \int_{I_m} f(x,y)\,dx = \int_{I_n} g_n(x)g_n(y)\,dx - \int_{I_{n+1}} g_{n+1}(x)g_n(y)\,dx \\ &= g_n(y)\int_{I_n} g_n(x)\,dx - g_n(y)\int_{I_{n+1}} g_{n+1}(x)\,dx \\ &= g_n(y) - g_n(y) = 0\end{aligned}$$

Here we have twice used that $\int_{I_m} g(t)\,dt = 1$ for all m. Thus $\int_0^1 f(x,y)\,dx = 0$ for all y and hence $\int_0^1 dy \Big[\int_0^1 dx\ f(x,y) \Big] = 0$.

Finally, fix any $0 \le x \le 1$ and let's compute $\int_0^1 f(x,y)\ dy$. That is, we are integrating f along a line that is parallel to the y–axis. If $x=0$, then $f(x,y)=0$ for all y, so $\int_0^1 f(x,y)\,dy = 0$. If $0 < x \le 1$, then there is exactly one positive integer m with $x \in I_m$. If $m \ge 2$, then $f(x,y)$ is zero, except for y in I_m and I_{m-1}. But, if $m=1$, then $f(x,y)$ is zero, except for y in I_1. (Take another look at the figure on the previous page.) So for $x \in I_m$, with $m \ge 2$,

$$\begin{aligned}\int_0^1 f(x,y)\,dy &= \sum_{n=m,m-1} \int_{I_n} f(x,y)\,dy = \int_{I_m} g_m(x) g_m(y)\,dy - \int_{I_{m-1}} g_m(x) g_{m-1}(y)\,dy \\ &= g_m(x) \int_{I_m} g_m(y)\,dy - g_m(x) \int_{I_{m-1}} g_{m-1}(y)\,dy \\ &= g_m(x) - g_m(x) = 0\end{aligned}$$

But $x \in I_1$,

$$\int_0^1 f(x,y)\,dy = \int_{I_1} f(x,y)\,dy = \int_{I_1} g_1(x) g_1(y)\,dy = g_1(x) \int_{I_1} g_1(y)\,dy = g_1(x)$$

Thus

$$\int_0^1 f(x,y)\,dy = \begin{cases} 0 & \text{if } x \le \delta_2 \\ g_1(x) & \text{if } x \in I_1 \end{cases}$$

and hence

$$\int_0^1 dx \Big[\int_0^1 dy\ f(x,y) \Big] = \int_{I_1} g_1(x)\,dx = 1$$

The conclusion is that for the $f(x,y)$ above, which is defined for all $0 \le x \le 1$ and $0 \le y \le 1$ and is continuous except at $(0,0)$,

$$\int_0^1 dy \Big[\int_0^1 dx\ f(x,y) \Big] = 0 \qquad \int_0^1 dx \Big[\int_0^1 dy\ f(x,y) \Big] = 1$$