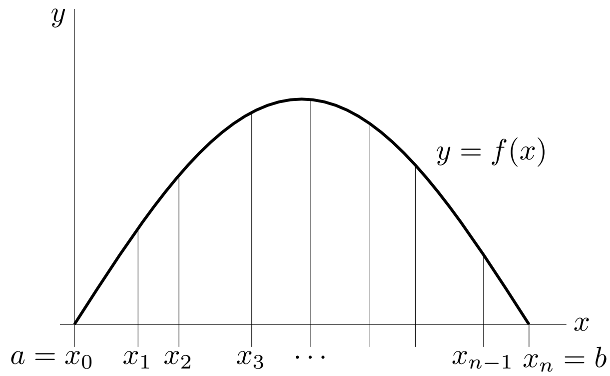
The Definition of the Integral in One Dimension

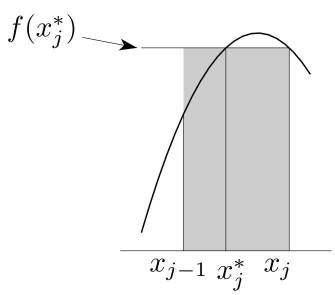
These notes discuss the definition of the definite integral $\int_a^b f(x)\,dx$. The integral is defined as the limit of a family of approximations to the area between the graph of $y = f(x)$ and the x–axis, with x running from a to b. To form these approximations, we select an integer n, we select $n+1$ values of x that obey

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and we also select an additional n values of x, denoted $x_1^*, x_2^*, \cdots, x_n^*$, that obey $x_{j-1} \le x_j^* \le x_j$ for all $1 \le j \le n$. The area between the graph of $y = f(x)$ and the x–axis, with x running from x_{j-1} to



x_j, i.e. the contribution of $\int_{x_{j-1}}^{x_j} f(x)\,dx$ to the integral, is approximated by the area of a rectangle. The rectangle has width $x_j - x_{j-1}$ and height $f(x_j^*)$.



Thus the approximation is

$$\int_a^b f(x)\,dx \approx f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

Of course every different choice of $\mathbb{P} = \big(n, x_1, x_2, \cdots, x_{n-1}, x_1^*, x_2^*, \cdots, x_n^*\big)$ gives a different approximation

$$\mathcal{I}(\mathbb{P}) = f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

But I claim that, for any reasonable function $f(x)$, if you take any sequence of these approximations, with the maximum width of the rectangles tending to zero, you always get exactly the same limiting value. This limiting value is defined to be $\int_a^b f(x)\,dx$. If we denote by

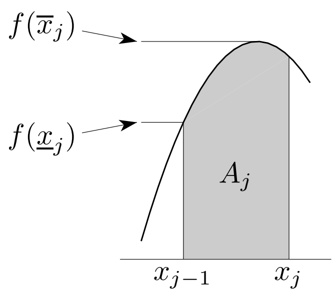
$$M(\mathbb{P}) = \max\big\{x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}\big\}$$

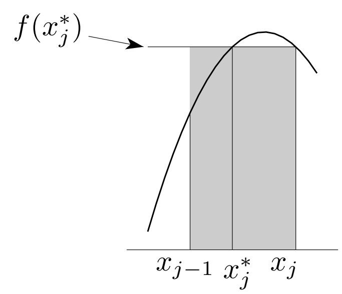
the maximum width of the rectangles used in the approximation determined by $\mathbb{P}$, then the definition of the definite integral is

$$\int_a^b f(x)\,dx = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P})$$

For the rest of these notes, assume that $f(x)$ is continuous for $a \le x \le b$, is differentiable for all $a < x < b$ and that $|f'(x)| \le F$, for some constant F. I will now show that, under these hypotheses, as $M(\mathbb{P})$ approachs zero, $\mathcal{I}(\mathbb{P})$ always approaches the the area, A, between the graph of $y = f(x)$ and the x–axis, with x running from a to b. These assumptions are chosen to make the argument particularly transparent. With a little more work one can weaken the hypotheses considerably. I am cheating a little by implicitly assuming that the area A exists. In fact, one can adjust the argument below to remove this implicit assumption.

Concentrate on A_j, the part of the area coming from $x_{j-1} \le x \le x_j$. We have approximated





this area by $f(x_j^*)[x_j - x_{j-1}]$. Let $f(\overline{x}_j)$ and $f(\underline{x}_j)$ be the largest and smallest values of $f(x)$ for $x_{j-1} \le x \le x_j$. The true area A_j has to lie somewhere between $f(\overline{x}_j)[x_j - x_{j-1}]$ and $f(\underline{x}_j)[x_j - x_{j-1}]$. As

$$\begin{aligned} f(\underline{x}_j)[x_j - x_{j-1}] \le {}& \quad A_j \quad \le f(\overline{x}_j)[x_j - x_{j-1}] \\ f(\underline{x}_j)[x_j - x_{j-1}] \le {}& f(x_j^*)[x_j - x_{j-1}] \le f(\overline{x}_j)[x_j - x_{j-1}] \end{aligned}$$

the error in this part of our approximation obeys

$$\big|A_j - f(x_j^*)[x_j - x_{j-1}]\big| \le [f(\overline{x}_j) - f(\underline{x}_j)][x_j - x_{j-1}]$$

By the Mean–Value Theorem, there exists a c between $\underline{x}_j$ and $\overline{x}_j$ such that

$$f(\overline{x}_j) - f(\underline{x}_j) = f'(c)[\overline{x}_j - \underline{x}_j]$$

By the assumption that $|f'(x)| \le F$ for all x and the fact that $\underline{x}_j$ and $\overline{x}_j$ must both be between x_{j-1} and x_j

$$\big|f(\overline{x}_j) - f(\underline{x}_j)\big| \le F\big|\overline{x}_j - \underline{x}_j\big| \le F[x_j - x_{j-1}]$$

Hence the error in this part of our approximation obeys

$$\big|A_j - f(x_j^*)[x_j - x_{j-1}]\big| \le F[x_j - x_{j-1}]^2$$

and the total error

$$\begin{aligned}\big|A-\mathcal{I}(\mathbb{P})\big| &= \Big|A-\sum_{j=1}^{n} f(x_j^*)[x_j-x_{j-1}]\Big| \le \sum_{j=1}^{n} \big|A_j - f(x_j^*)[x_j-x_{j-1}]\big| \\ &\le \sum_{j=1}^{n} F[x_j-x_{j-1}]^2 \qquad\qquad = \sum_{j=1}^{n} F[x_j-x_{j-1}][x_j-x_{j-1}] \\ &\le \sum_{j=1}^{n} FM(\mathbb{P})[x_j-x_{j-1}] \qquad = FM(\mathbb{P})\sum_{j=1}^{n}[x_j-x_{j-1}] \\ &= FM(\mathbb{P})\ (b-a)\end{aligned}$$

Since a, b and F are fixed, this tends to zero as the maximum rectangle width $M(\mathbb{P})$ tends to zero, as desired.

