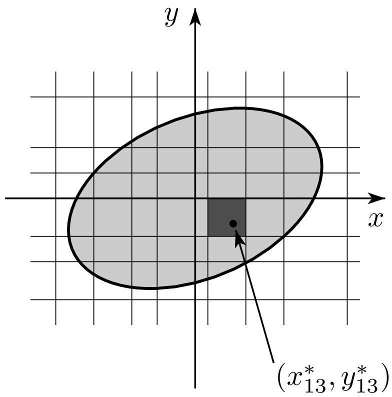
The Definition of the Integral in Two Dimensions

The integral $\iint_R f(x,y)\,dx\,dy$, where R is a bounded region in $\mathbb{R}^2$, is defined as follows.



- subdivide R by drawing lines parallel to the x and y axes
- number the resulting rectangles contained in R, 1 through n
- denote by ΔA_i the area of rectangle $\#i$
- select an arbitrary point (x^*_i, y^*_i) in rectangle $\#i$
- form the sum $\sum\limits_{i=1}^{n} f(x^*_i, y^*_i)\Delta A_i$

Now repeat this construction over and over again, using finer and finer grids. If, as the maximum diagonal of the rectangles approachs zero, this sum approachs a unique limit (independent of the choice of parallel lines and of points (x^*_i, y^*_i)), then

$$\iint_R f(x,y)\;dx\,dy = \lim \sum_{i=1}^{n} f(x^*_i, y^*_i)\,\Delta A_i$$

Theorem. *If $f(x,y)$ is continuous in a region R described by*

$$x_1 \le x \le x_2$$
$$y_1(x) \le y \le y_2(x)$$

for continuous functions $y_1(x)$, $y_2(x)$, then

$$\iint_R f(x,y)\;dx\,dy \qquad \textit{and} \qquad \int_{x_1}^{x_2} dx\left[\int_{y_1(x)}^{y_2(x)} dy\;f(x,y)\right]$$

both exist and are equal. Similarly, if R is described by

$$y_1 \le y \le y_2$$
$$x_1(y) \le x \le x_2(y)$$

for continuous functions $x_1(y)$, $x_2(y)$, then

$$\iint_R f(x,y)\;dx\,dy \qquad \textit{and} \qquad \int_{y_1}^{y_2} dy\left[\int_{x_1(y)}^{x_2(y)} dx\;f(x,y)\right]$$

both exist and are equal.

The proof of this theorem is not particularly difficult. But we still do not have time to go through it. The main ideas in the proof can already be seen in the notes “The Definition of the Integral in One Dimension”. An important consequence of this theorem is

Theorem (Fubini) *If $f(x,y)$ is continuous in a region R described by both*

$$x_1 \le x \le x_2$$
$$y_1(x) \le y \le y_2(x)$$

and

$$y_1 \le y \le y_2$$
$$x_1(y) \le x \le x_2(y)$$

for continuous functions $y_1(x)$, $y_2(x)$, $x_1(y)$, $x_2(y)$, then both

$$\int_{x_1}^{x_2} dx \left[\int_{y_1(x)}^{y_2(x)} dy \ f(x,y) \right] \qquad \text{and} \qquad \int_{y_1}^{y_2} dy \left[\int_{x_1(y)}^{x_2(y)} dx \ f(x,y) \right]$$

exist and are equal.

The hypotheses of both of these theorems can be relaxed a bit, but not too much. For example, if

$$R = \left\{ \ (x,y) \ \middle| \ 0 \le x \le 1, \ 0 \le y \le 1 \ \right\} \qquad f(x,y) = \begin{cases} 1 & \text{if } x, \ y \text{ are both rational numbers} \\ 0 & \text{otherwise} \end{cases}$$

then the integral $\iint_R f(x,y) \ dx\,dy$ does not exist. This is easy to see. If all of the x_i^*'s and y_i^*'s happen to be rational numbers, then

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \, \Delta A_i = \sum_{i=1}^{n} \Delta A_i = \text{Area of } R = 1$$

But if all of the x_i^*'s and y_i^*'s happen to be irrational numbers, then

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \, \Delta A_i = \sum_{i=1}^{n} 0 \, \Delta A_i = 0$$

So the limit of $\sum_{i=1}^{n} f(x_i^*, y_i^*) \, \Delta A_i$, as the maximum diagonal of the rectangles approachs zero, depends on the choice of points (x_i^*, y_i^*). So the integral $\iint_R f(x,y) \ dx\,dy$ does not exist.

The notes "A Fubini Counterexample" contain an even more pathological example. In those notes, we relax exactly one of the hypotheses of Fubini's Theorem, namely the continuity of f, and construct an example in which both of the integrals in Fubini's Theorem exist, but are **not equal**. In fact, we choose $R = \left\{ \ (x,y) \ \middle| \ 0 \le x \le 1, \ 0 \le y \le 1 \ \right\}$ and we use a function $f(x,y)$ that is continuous on R, except at exactly one point – the origin.