Limits

Notation.

- $\mathbb{N}$ is the set $\{1, 2, 3, \cdots\}$ of all natural numbers
- $\mathbb{R}$ is the set of all real numbers
- $\forall$ is read "for all"
- $\exists$ is read "there exists"
- $\in$ is read "element of"
- $\notin$ is read "not an element of"
- $\{\ A \mid B\ \}$ is read "the set of all A such that B"
- If S is a set and T is a subset of S, then $S \setminus T$ is $\{\ x \in S \mid x \notin T\ \}$, the set S with the elements of T removed.
- if n is a natural number, $\mathbb{R}^n$ is used for both the set of n–component vectors $\langle x_1, x_2, \cdots, x_n \rangle$ and the set of points $(x_1, x_2, \cdots, x_n)$ with n–coordinates.
- If S and T are sets, then $f : S \to T$ means that f is a function which assigns to each element of S an element of T.
- $[a, b] = \{\ x \in \mathbb{R} \mid a \le x \le b\ \}$
  $(a, b] = \{\ x \in \mathbb{R} \mid a < x \le b\ \}$
  $[a, b) = \{\ x \in \mathbb{R} \mid a \le x < b\ \}$
  $(a, b) = \{\ x \in \mathbb{R} \mid a < x < b\ \}$

Roughly speaking, $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}$ means that $\vec{f}(\vec{x})$ approachs $\vec{L}$ as $\vec{x}$ approachs $\vec{a}$. Here is the precise definition of limit, and a couple of related definitions.

Definition 1 Let $m, n \in \mathbb{N}$.

(a) Let $\vec{a} \in \mathbb{R}^n$ and $\vec{L} \in \mathbb{R}^m$, and let $\vec{f} : \mathbb{R}^n \setminus \{\vec{a}\} \to \mathbb{R}^m$. Then $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \quad \big|\vec{f}(\vec{x}) - \vec{L}\big| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

(b) Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous at $\vec{a} \in \mathbb{R}^n$ if $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$ and $\vec{f}$ is continous on $\mathbb{R}^n$ if it is continuous at every $\vec{a} \in \mathbb{R}^n$.

Remark 2

(a) Here is what that definition of limit says. Suppose you have a magic microscope whose magnification can be set as high as you like. Suppose that when the magnification is set to $\frac{1}{\varepsilon}$, you can only see those points whose distance from $\vec{L}$ is less than ε. The definition says that no matter how high you set the magnification, (i.e. no matter how small you set $\varepsilon > 0$), you will be able to see $\vec{f}(\vec{x})$ whenever $\vec{x}$ is close enough to $\vec{a}$ (if the distance from $\vec{x}$ to $\vec{a}$ is less than δ, then you will certainly see $\vec{f}(\vec{x})$).

(b) Definition 1.a, of $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$, is set up so that the function $\vec{f}(\vec{x})$ is never evaluated at $\vec{x} = \vec{a}$. Indeed $\vec{f}(\vec{x})$ need not even be defined at $\vec{x} = \vec{a}$. This is exactly what happens in the definition of the derivative $h'(a) = \lim_{x\to a} \frac{h(x)-h(a)}{x-a}$. (In this case $f(x) = \frac{h(x)-h(a)}{x-a}$.)

We'll first do a couple of examples with $m = n = 1$. We'll do higher dimensional examples later.

Example 3 In Example 2 of the notes "A Little Logic" we saw that the statement

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \quad \text{if } |x| < \delta \text{ then } x^2 < \varepsilon$$

is true. Consequently

$$\lim_{x\to 0} x^2 = 0$$

Example 4 In this example, we consider $\lim_{x\to 0} \sin\frac{1}{x}$. So fix any real number L and let

- $S(\delta,\varepsilon)$ be the statement "$|\sin\frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$",
- $T(\varepsilon)$ be the statement "$\exists\, \delta > 0$ such that $S(\delta,\varepsilon)$" or

$$\exists\, \delta > 0 \ \text{ such that } \ |\sin\tfrac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

- U be the statement "$\forall\, \varepsilon > 0 \ \ T(\varepsilon)$" or

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \ |\sin\tfrac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

Then

- Fix any $\varepsilon > 0$ and any $\delta > 0$. The statement $S(\delta,\varepsilon)$ is true if all values of $\sin\frac{1}{x}$, with $0 < |x| < \delta$, lie in the interval $(L-\varepsilon, L+\varepsilon)$. As x runs over the interval $(0,\delta)$, (so that, in particular, $0 < |x| < \delta$) $\frac{1}{x}$ covers the set $\left(\frac{1}{\delta}, \infty\right)$. This contains many intervals of length 2π and hence many periods of sin. So, as x runs over the interval $(0,\delta)$, $\sin\frac{1}{x}$ covers all of $[-1,1]$. So $S(\delta,\varepsilon)$ is true if and only if the interval $[-1,1]$ is contained in the interval $(L-\varepsilon, L+\varepsilon)$. In particular, when $\varepsilon < 1$, the interval $(L-\varepsilon, L+\varepsilon)$, which has length 2ε, is shorter than $[-1,1]$ and cannot contain it, so that $S(\delta,\varepsilon)$ is false.
- Because $S(\delta,\varepsilon)$is false for all $\delta > 0$ when $\varepsilon < 1$, $T(\varepsilon)$ is false for all $\varepsilon < 1$.
- U is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T\left(\frac{1}{2}\right)$ is false.

In conclusion, $\sin\frac{1}{x}$ has no limit as $x \to 0$.

Theorem 5 *Let* $n \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathbb{R}^n$, $F, G \in \mathbb{R}$ *and*

$$f, g : \mathbb{R}^n \setminus \{\vec{a}\} \to \mathbb{R} \qquad \vec{X} : \mathbb{R}^n \setminus \{\vec{b}\} \to \mathbb{R}^n \setminus \{\vec{a}\} \qquad \gamma : \mathbb{R} \to \mathbb{R}$$

Assume that

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = F \qquad \lim_{\vec{x}\to\vec{a}} g(\vec{x}) = G \qquad \lim_{\vec{y}\to\vec{b}} \vec{X}(\vec{y}) = \vec{a} \qquad \lim_{t\to F} \gamma(t) = \gamma(F) = \Gamma$$

Then

(*a*) $\displaystyle\lim_{\vec{x}\to\vec{a}} \big[f(\vec{x}) + g(\vec{x})\big] = F + G$

(*b*) $\displaystyle\lim_{\vec{x}\to\vec{a}} f(\vec{x})g(\vec{x}) = FG$

(*c*) $\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{F}{G} \qquad$ *if* $G \neq 0$

(*d*) $\displaystyle\lim_{\vec{y}\to\vec{b}} f\big(\vec{X}(\vec{y})\big) = F$

(*e*) $\displaystyle\lim_{\vec{x}\to\vec{a}} \gamma\big(f(\vec{x})\big) = \Gamma$

Proof: Note that the ε and δ in "$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0$ such that $S(\delta, \varepsilon)$" are dummy variables, just as x is a dummy variable in $\int_0^1 x\,dx$. You may replace ε and δ by whatever symbols you like. The hypotheses of this theorem say that

$$\forall\,\varepsilon_f > 0\ \ \exists\,\delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_f \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f \tag{1}$$
$$\forall\,\varepsilon_g > 0\ \ \exists\,\delta_g > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_g \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_g \tag{2}$$
$$\forall\,\varepsilon_X > 0\ \ \exists\,\delta_X > 0 \text{ such that } |X(\vec{y}) - \vec{a}| < \varepsilon_X \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X \tag{3}$$
$$\forall\,\varepsilon_\gamma > 0\ \ \exists\,\delta_\gamma > 0 \text{ such that } |\gamma(t) - \Gamma| < \varepsilon_\gamma \text{ whenever } 0 < |t - F| < \delta_\gamma \tag{4}$$

(a) We are to prove that

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0 \text{ such that } |f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

Observe that

$$|f(\vec{x}) + g(\vec{x}) - F - G| = \big|[f(\vec{x}) - F] + [g(\vec{x}) - G]\big| \le |f(\vec{x}) - F| + |g(\vec{x}) - G|$$

Set $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. By (1) with $\varepsilon_f = \varepsilon_1$ and (2) with $\varepsilon_g = \varepsilon_2$,

$$\exists\, \delta_1 > 0 \ \text{ such that } |f(\vec{x}) - F| < \varepsilon_1 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_1$$
$$\exists\, \delta_2 > 0 \ \text{ such that } |g(\vec{x}) - G| < \varepsilon_2 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_2$$

Choose $\delta = \min\big\{\delta_1, \delta_2\big\}$. Then whenever $0 < |\vec{x} - \vec{a}| < \delta$ we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ so that

$$|f(\vec{x}) + g(\vec{x}) - F - G| \le |f(\vec{x}) - F| + |g(\vec{x}) - G| < \varepsilon_1 + \varepsilon_2 = \varepsilon$$

(b) is a homework assignment.

(c) We are to prove that

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \left|\tfrac{f(\vec{x})}{g(\vec{x})} - \tfrac{F}{G}\right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\left|\tfrac{f(\vec{x})}{g(\vec{x})} - \tfrac{F}{G}\right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

Set $\varepsilon_1 = \frac{1}{6}|G|\varepsilon$ and $\varepsilon_2 = \frac{G^2}{6(|F|+1)}\varepsilon$. By (1) with $\varepsilon_f = \varepsilon_1$, (2) with $\varepsilon_g = \varepsilon_2$ and (2) with $\varepsilon_g = \frac{1}{2}|G|$,

$$\exists\, \delta_1 > 0 \ \text{ such that } |f(\vec{x}) - F| < \varepsilon_1 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_1$$
$$\exists\, \delta_2 > 0 \ \text{ such that } |g(\vec{x}) - G| < \varepsilon_2 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_2$$
$$\exists\, \delta_3 > 0 \ \text{ such that } |g(\vec{x}) - G| < \tfrac{1}{2}|G| \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_3$$

Choose $\delta = \min\big\{\delta_1, \delta_2, \delta_3\big\}$. Then whenever $0 < |\vec{x} - \vec{a}| < \delta$ we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ and $0 < |\vec{x} - \vec{a}| < \delta_3$ so that

$$\begin{aligned}
\left|\frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G}\right| &= \frac{|f(\vec{x})G - Fg(\vec{x})|}{|g(\vec{x})G|} = \frac{|\{f(\vec{x}) - F\}G - F\{g(\vec{x}) - G\}|}{|g(\vec{x})G|} \\
&\le \frac{|f(\vec{x}) - F|\,|G| + |F|\,|g(\vec{x}) - G|}{|g(\vec{x})|\,|G|} \\
&\le \frac{\varepsilon_1\,|G| + |F|\,\varepsilon_2}{\frac{1}{2}|G|\,|G|} \quad \text{since } |g(\vec{x})| = \big|g(\vec{x}) - G + G\big| \ge |G| - |g(\vec{x}) - G| \ge \tfrac{1}{2}|G| \\
&= \tfrac{1}{6}|G|\varepsilon\ \tfrac{|G|}{G^2/2} + \tfrac{|F|}{G^2/2}\tfrac{G^2}{6(|F|+1)}\varepsilon = \tfrac{\varepsilon}{3} + \tfrac{1}{3}\tfrac{|F|}{|F|+1}\varepsilon \\
&< \varepsilon
\end{aligned}$$

(d) We are to prove that

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \left|f\big(\vec{X}(\vec{y})\big) - F\right| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\left|f\big(\vec{X}(\vec{y})\big) - F\right| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta$$

By (1) with $\varepsilon_f = \varepsilon$

$$\exists\, \delta_f > 0 \ \text{ such that } |f(\vec{x}) - F| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

and (3) with $\varepsilon_X = \delta_f$,

$$\exists\, \delta_X > 0 \ \text{ such that } |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X$$

Choosing $\delta = \delta_X$, we have

$$0 < |\vec{y} - \vec{b}| < \delta = \delta_X \implies 0 < |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \implies \left|f\big(\vec{X}(\vec{y})\big) - F\right| < \varepsilon$$

(e) has essentially the same proof as part (d). We are to prove that

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \left|\gamma\big(f(\vec{x})\big) - \Gamma\right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\left|\gamma\big(f(\vec{x})\big) - \Gamma\right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

By (4) with $\varepsilon_\gamma = \varepsilon$ and the hypothesis that $\gamma(F) = \Gamma$

$$\exists\, \delta_\gamma > 0 \ \text{ such that } |\gamma(t) - \Gamma| < \varepsilon \text{ whenever } |t - F| < \delta_\gamma$$

By (1) with $\varepsilon_f = \delta_\gamma$

$$\exists\, \delta_f > 0 \ \text{ such that } |f(\vec{x}) - F| < \delta_\gamma \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

Choosing $\delta = \delta_f$, we have

$$0 < |\vec{x} - \vec{a}| < \delta = \delta_f \implies \left|f(\vec{x}) - F\right| < \delta_\gamma \implies \left|\gamma\big(f(\vec{x})\big) - \Gamma\right| < \varepsilon$$

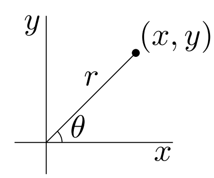
■

Example 6 There is a typical application of Theorem 5. Here " $\overset{a}{=}$ " means that Theorem 5.a justifies that equality.

$$\begin{aligned}
\lim_{(x,y)\to(2,3)} \big(x+\sin y\big) &\overset{a}{=} \lim_{(x,y)\to(2,3)} x + \lim_{(x,y)\to(2,3)} \sin y \\
&\overset{e}{=} \lim_{(x,y)\to(2,3)} x + \sin\Big(\lim_{(x,y)\to(2,3)} y\Big) \\
&= 2+\sin 3 \\
\lim_{(x,y)\to(2,3)} \big(x^2y^2+1\big) &\overset{a}{=} \lim_{(x,y)\to(2,3)} x^2y^2 + \lim_{(x,y)\to(2,3)} 1 \\
&\overset{b}{=} \Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} x\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big)\Big(\lim_{(x,y)\to(2,3)} y\Big) + 1 \\
&= 2^2 3^2 + 1 \\
\lim_{(x,y)\to(2,3)} \frac{x+\sin y}{x^2y^2+1} &\overset{c}{=} \frac{\lim_{(x,y)\to(2,3)}(x+\sin y)}{\lim_{(x,y)\to(2,3)}(x^2y^2+1)} \\
&= \frac{2+\sin 3}{37}
\end{aligned}$$

Here we have used that $\sin x$ is a continuous function. In this course we shall assume that we already know that "standard single variable calculus functions" like $\sin x$, $\cos x$, e^x and so on are continuous.

Example 7 As a second example, we consider $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$. In this example, both the numerator, x^2y, and the denominator x^2+y^2 tend to zero as (x,y) approachs $(0,0)$, so we have to be more careful. A good way to see the behaviour of a function $f(x,y)$ when (x,y) is close to $(0,0)$ is to switch to the polar coordinates r,θ using



$$x = r\cos\theta$$
$$y = r\sin\theta$$

Recall that the definition of $\lim_{(x,y)\to(0,0)} f(x,y) = L$ is

$$\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \text{ such that } \big|f(x,y)-L\big|<\varepsilon \text{ whenever } 0<|(x,y)|<\delta \qquad (5)$$

The condition $0<|(x,y)|<\delta$ says that $0<r<\delta$ and no restriction on θ. So substituting $x=r\cos\theta$, $y=r\sin\theta$ into (5) gives

$$\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \text{ such that } \big|f(r\cos\theta,r\sin\theta)-L\big|<\varepsilon \text{ whenever } 0<r<\delta,\ 0\le\theta\le 2\pi \quad (6)$$

For our current example

$$\tfrac{x^2y}{x^2+y^2} = \tfrac{(r\cos\theta)^2(r\sin\theta)}{r^2} = r\cos^2\theta\sin\theta$$

As $\big|r\cos^2\theta\sin\theta\big|\le r\to 0$ when $r\to 0$, we have

$$\lim_{(x,y)\to(0,0)} \tfrac{x^2y}{x^2+y^2} = 0$$

Example 8 As a third example, we consider $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. Once again, the best way to see the behaviour of $\frac{x^2-y^2}{x^2+y^2}$ for (x,y) close to $(0,0)$ is to switch to polar coordinates.

$$\frac{x^2-y^2}{x^2+y^2} = \frac{(r\cos\theta)^2-(r\sin\theta)^2}{r^2} = \cos^2\theta - \sin^2\theta = \cos(2\theta)$$

No matter how small you make $\delta > 0$, as (x,y) runs over those points with $r = |(x,y)| < \delta$, $\frac{x^2-y^2}{x^2+y^2}$ takes all values in the interval $[-1,1]$. So $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.