A Little Logic

"There Exists" and "For All"

The symbol $\exists$ is read "there exists" and the symbol $\forall$ is read "for all" (or "for each" or "for every", if it reads better). Let $S(\varepsilon)$ be a statement that contains the parameter ε. For example, $S(\varepsilon)$ might be " $5 < \varepsilon$". Then

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is true if there exists at least one $\varepsilon > 0$ such that $S(\varepsilon)$ is true and
- the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is true if $S(\varepsilon)$ is true whenever $\varepsilon > 0$.

On the other hand

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is false when $S(\varepsilon)$ is false for every $\varepsilon > 0$ and
- the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is false when there exists at least one $\varepsilon > 0$ for which $S(\varepsilon)$ is false.

Example 1 Let $S(\varepsilon)$ be the statement " $5 < \varepsilon$". Then

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is true since there does indeed exist an $\varepsilon > 0$, for example $\varepsilon = 6$, such that $S(\varepsilon)$ is true.
- On the other hand, the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is false since there is at least one $\varepsilon > 0$, for example $\varepsilon = 4$, such that $S(\varepsilon)$ is is false.

Let $S(\delta, \varepsilon)$ be a statement that contains the two parameters δ and ε. For example, $S(\delta, \varepsilon)$ might be "if $|x| < \delta$ then $x^2 < \varepsilon$". Define the statement U to be

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } S(\delta, \varepsilon)$$

To analyse U, define, for each $\varepsilon > 0$, the statement $T(\varepsilon)$ to be "$\exists\, \delta > 0$ such that $S(\delta, \varepsilon)$". Then U is the statement "$\forall\, \varepsilon > 0 \ \ T(\varepsilon)$" and

- U is true if $T(\varepsilon)$ is true for every $\varepsilon > 0$.
- Given any fixed $\varepsilon_0 > 0$, $T(\varepsilon_0)$ is true if there exists at least one $\delta > 0$ such that $S(\delta, \varepsilon_0)$ is true.
- So, all together, U is true if for each $\varepsilon > 0$, there exists at least one $\delta > 0$ (which may depend on ε) such that $S(\delta, \varepsilon)$ is true.

On the other hand

- U is false if $T(\varepsilon)$ is false for at least one $\varepsilon > 0$.
- Given any fixed $\varepsilon_0 > 0$, $T(\varepsilon_0)$ is false if there does not exist at least one $\delta > 0$ such that $S(\delta, \varepsilon_0)$ is true. That is, if $S(\delta, \varepsilon_0)$ is false for all $\delta > 0$.

- So, all together, U is false if there exists at least one $\varepsilon > 0$, such that $S(\delta, \varepsilon)$ is false for all $\delta > 0$. That is, U is false if the statement

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, \delta > 0\ S(\delta, \varepsilon) \text{ is false}$$

is true.

Example 2 In this example, we will always assume that $\delta > 0$ and $\varepsilon > 0$. Let

- $S(\delta, \varepsilon)$ be the statement "if $|x| < \delta$ then $x^2 < \varepsilon$",
- $T(\varepsilon)$ be the statement "$\exists\, \delta > 0$ such that $S(\delta, \varepsilon)$" or

$$\exists\, \delta > 0 \ \text{ such that } \quad \text{if } |x| < \delta \text{ then } x^2 < \varepsilon$$

- U be the statement "$\forall\, \varepsilon > 0\ \ T(\varepsilon)$" or

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \quad \text{if } |x| < \delta \text{ then } x^2 < \varepsilon$$

Then

- For example, $S(\delta, \varepsilon)$ is true when $\delta = 2$ and $\varepsilon = 2^2 = 4$. That is $S(2, 4)$ is true. On the other hand $S(2, 3)$ is false. In general, $S(\delta, \varepsilon)$ is true if and only if $\varepsilon \geq \delta^2$, because as x runs over the interval $-\delta < x < \delta$, x^2 covers the set $0 \leq x^2 < \delta^2$.
- For example, $T(4)$ is true because when $\varepsilon = 4$, we may choose $\delta = 2$ and then $S(\delta = 2, \varepsilon = 4)$ is true. In fact, $T(\varepsilon)$ is true for every $\varepsilon > 0$, because we may choose $\delta = \sqrt{\varepsilon}$ and then $S\big(\sqrt{\varepsilon}, \varepsilon\big)$ is true.
- U is true since, as we have just seen, $T(\varepsilon)$ is true for all $\varepsilon > 0$.

Example 3 In this example, we will again assume that $\delta > 0$ and $\varepsilon > 0$. Let

- $S(\delta, \varepsilon)$ be the statement "if $|x| < \delta$ then $1 + x^2 < \varepsilon$",
- $T(\varepsilon)$ be the statement "$\exists\, \delta > 0$ such that $S(\delta, \varepsilon)$" or

$$\exists\, \delta > 0 \ \text{ such that } \quad \text{if } |x| < \delta \text{ then } 1 + x^2 < \varepsilon$$

- U be the statement "$\forall\, \varepsilon > 0\ \ T(\varepsilon)$" or

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \quad \text{if } |x| < \delta \text{ then } 1 + x^2 < \varepsilon$$

Then

- when x runs over the interval $-\delta < x < \delta$, $1 + x^2$ covers the set $1 \leq 1 + x^2 < 1 + \delta^2$. Hence $S(\delta, \varepsilon)$ is true if and only if $\varepsilon \geq 1 + \delta^2$.

- Because $S(\delta, \varepsilon)$ is true if and only if $\varepsilon \geq 1 + \delta^2$, the statement $T(\varepsilon)$ is equivalent to "$\exists\, \delta > 0$ such that $\varepsilon \geq 1 + \delta^2$" which is true if and only if $\varepsilon > 1$. (If $\varepsilon > 1$, we may choose $\delta = \sqrt{\varepsilon - 1}$. If $\varepsilon < 1$, no δ works since $1 + \delta^2$ is always at least 1. If $\varepsilon = 1$, the only δ which could work is $\delta = 0$, and it does not satisfy the condition $\delta > 0$.)
- U is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T\left(\frac{1}{2}\right)$ is false.

Converse, Inverse, Contrapositive

Let S_1 and S_2 be statements. For example S_1 might be "x is a rational number" and S_2 might be "x is a real number". Define the statement T to be "If S_1 is true then S_2 is true.". Then

- the converse of T is the statement "If S_2 is true then S_1 is true.",
- the inverse of T is the statement "If S_1 is false then S_2 is false." and
- the contrapositive of T is the statement "If S_2 is false then S_1 is false."

If the statement T is true, then

- the converse of T need not be true,
- the inverse of T need not be true and
- the contrapositive of T is necessarily true.

Example 4 Let S_1 be the statement "x is a rational number" and S_2 be the statement "x is a real number". Then

- T is the statement "If x is a rational number then x is a real number." and is true,
- the converse of T is the statement "If x is a real number then x is a rational number." and is false,
- the inverse of T is the statement "If x is not a rational number then x is not a real number." and is false, and
- the contrapositive of T is the statement "If x is not a real number then x is not a rational number." and is true.