Equality of Mixed Partials

Theorem. *If the partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *exist and are continuous at* (x_0, y_0)*, then*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Proof: Here is an outline of the proof. The details are given as footnotes at the end of the outline. Fix x_0 and y_0 and define$^{(1)}$

$$F(h,k) = \tfrac{1}{hk}\big[f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)\big]$$

Then, by the mean value theorem,

$$\begin{aligned}
F(h,k) &\overset{2}{=} \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) - \tfrac{\partial f}{\partial y}(x_0, y_0+\theta_1 k)\Big] \\
&\overset{3}{=} \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) \\
F(h,k) &\overset{4}{=} \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+k) - \tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0)\Big] \\
&\overset{5}{=} \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)
\end{aligned}$$

for some $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$. All of $\theta_1, \theta_2, \theta_3, \theta_4$ depend on x_0, y_0, h, k. Hence

$$\tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)$$

for all h and k. Taking the limit $(h,k) \to (0,0)$ and using the assumed continuity of both partial derivatives at (x_0, y_0) gives

$$\tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0, y_0) = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0, y_0)$$

The Details

(1) We define $F(h,k)$ in this way because both partial derivatives $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ are defined as limits of $F(h,k)$ as $h, k \to 0$. For example,

$$\begin{aligned}
&\lim_{h\to 0} F(h,k) = \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0, y_0+k) - \tfrac{\partial f}{\partial x}(x_0, y_0)\Big] \\
\Longrightarrow \qquad &\lim_{k\to 0}\lim_{h\to 0} F(h,k) = \lim_{k\to 0} \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0, y_0+k) - \tfrac{\partial f}{\partial x}(x_0, y_0)\Big] \\
&\qquad = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0, y_0)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{k\to 0} F(h,k) &= \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h,y_0) - \tfrac{\partial f}{\partial y}(x_0,y_0)\Big] \\
\implies \qquad \lim_{h\to 0}\lim_{k\to 0} F(h,k) &= \lim_{h\to 0} \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h,y_0) - \tfrac{\partial f}{\partial y}(x_0,y_0)\Big] \\
&= \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0,y_0)
\end{aligned}$$

(2) Define $G(y) = f(x_0+h,y) - f(x_0,y)$. By the mean value theorem

$$\begin{aligned}
F(h,k) &= \tfrac{1}{h}\Big[\tfrac{G(y_0+k)-G(y_0)}{k}\Big] \\
&= \tfrac{1}{h}\tfrac{dG}{dy}(y_0+\theta_1 k) \qquad \text{for some } 0<\theta_1<1 \\
&= \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h,y_0+\theta_1 k) - \tfrac{\partial f}{\partial y}(x_0,y_0+\theta_1 k)\Big]
\end{aligned}$$

(3) Define $H(x) = \frac{\partial f}{\partial y}(x, y_0+\theta_1 k)$. By the mean value theorem

$$\begin{aligned}
F(h,k) &= \tfrac{1}{h}\Big[H(x_0+h) - H(x_0)\Big] \\
&= \tfrac{dH}{dx}(x_0+\theta_2 h) \qquad \text{for some } 0<\theta_2<1 \\
&= \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k)
\end{aligned}$$

(4) Define $A(x) = f(x,y_0+k) - f(x,y_0)$. By the mean value theorem

$$\begin{aligned}
F(h,k) &= \tfrac{1}{k}\Big[\tfrac{A(x_0+h)-A(x_0)}{h}\Big] \\
&= \tfrac{1}{k}\tfrac{dA}{dx}(x_0+\theta_3 h) \qquad \text{for some } 0<\theta_3<1 \\
&= \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0+\theta_3 h,y_0+k) - \tfrac{\partial f}{\partial x}(x_0+\theta_3 h,y_0)\Big]
\end{aligned}$$

(5) Define $B(y) = \frac{\partial f}{\partial x}(x_0+\theta_3 h, y)$. By the mean value theorem

$$\begin{aligned}
F(h,k) &= \tfrac{1}{k}\Big[B(y_0+k) - B(y_0)\Big] \\
&= \tfrac{dB}{dy}(y_0+\theta_4 k) \qquad \text{for some } 0<\theta_4<1 \\
&= \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)
\end{aligned}$$

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Example. Here is an example which shows that it is **not always true** that $\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)$. Define

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

This function is continuous everywhere. We now compute the first order partial derivatives. For $(x, y) \neq (0,0)$

$$\frac{\partial f}{\partial x}(x,y) = y\tfrac{x^2-y^2}{x^2+y^2} + xy\tfrac{2x}{x^2+y^2} - xy\tfrac{2x(x^2-y^2)}{(x^2+y^2)^2} = y\tfrac{x^2-y^2}{x^2+y^2} + xy\tfrac{4xy^2}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial y}(x,y) = x\tfrac{x^2-y^2}{x^2+y^2} - xy\tfrac{2y}{x^2+y^2} - xy\tfrac{2y(x^2-y^2)}{(x^2+y^2)^2} = x\tfrac{x^2-y^2}{x^2+y^2} - xy\tfrac{4yx^2}{(x^2+y^2)^2}$$

For $(x, y) = (0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \left[\tfrac{d}{dx} f(x,0)\right]_{x=0} = \left[\tfrac{d}{dx} 0\right]_{x=0} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \left[\tfrac{d}{dy} f(0,y)\right]_{y=0} = \left[\tfrac{d}{dy} 0\right]_{y=0} = 0$$

By way of summary, the two first order partial derivatives are

$$f_x(x,y) = \begin{cases} y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
$$f_y(x,y) = \begin{cases} x\frac{x^2-y^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Both $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous. Finally, we compute

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \left[\tfrac{d}{dx} f_y(x,0)\right]_{x=0} = \lim_{h\to 0} \tfrac{1}{h}\left[f_y(h,0) - f_y(0,0)\right] = \lim_{h\to 0} \tfrac{1}{h}\left[h\tfrac{h^2-0^2}{h^2+0^2} - 0\right] = 1$$
$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \left[\tfrac{d}{dy} f_x(0,y)\right]_{y=0} = \lim_{k\to 0} \tfrac{1}{k}\left[f_x(0,k) - f_x(0,0)\right] = \lim_{k\to 0} \tfrac{1}{k}\left[k\tfrac{0^2-k^2}{0^2+k^2} - 0\right] = -1$$