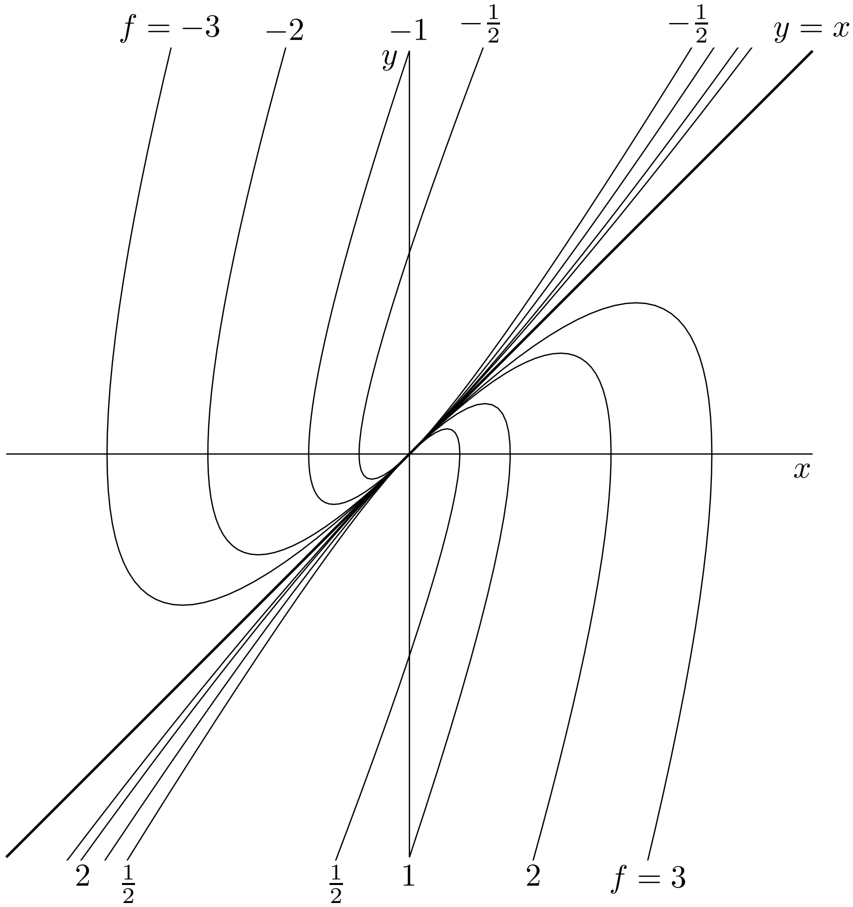
A Limit That Doesn't Exist

In this example we study the behaviour of the function

$$f(x,y) = \begin{cases} \frac{(2x-y)^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

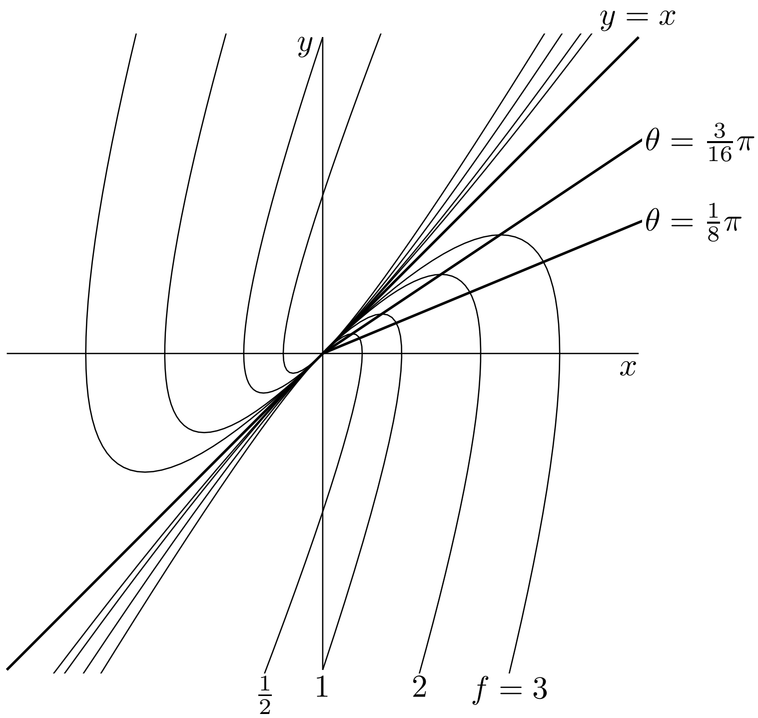
as $(x,y) \to (0,0)$. Here is a graph of the level curves, $f(x,y) = c$, of this function for various values of the constant c.



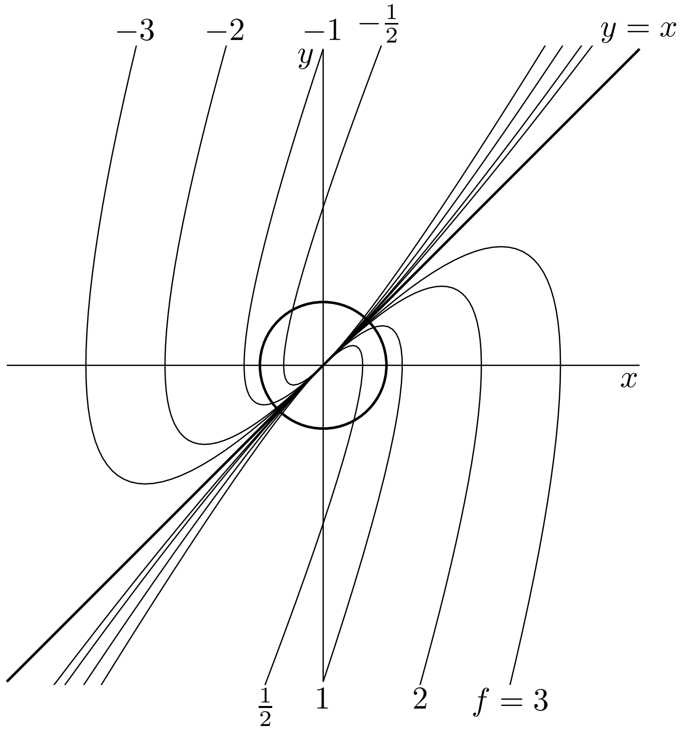
In polar coordinates

$$f(r\cos\theta, r\sin\theta) = \begin{cases} r\frac{(2\cos\theta - \sin\theta)^2}{\cos\theta - \sin\theta} & \text{if } \cos\theta \neq \sin\theta \\ 0 & \text{if } \cos\theta = \sin\theta \end{cases}$$

If we approach the origin along any fixed ray $\theta = \text{const}$, then $f(r\cos\theta, r\sin\theta)$ is the constant $\frac{(2\cos\theta-\sin\theta)^2}{\cos\theta-\sin\theta}$ (or 0 if $\cos\theta = \sin\theta$) times r and so approachs zero as r approachs zero. You can see this in the figure on the left below, which shows the level curves again, with the rays $\theta = \frac{1}{8}\pi$ and $\theta = \frac{3}{16}\pi$ superimposed.



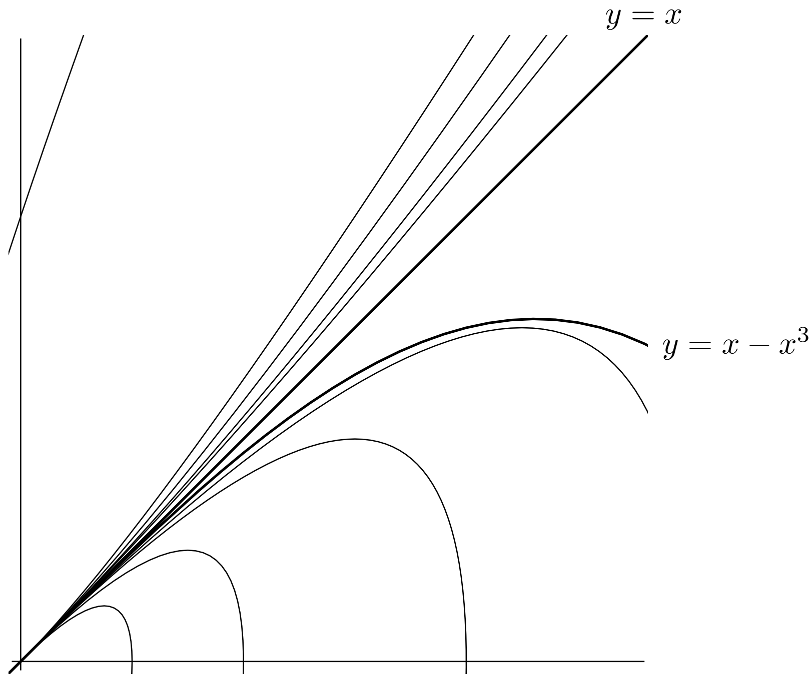
Nontheless, $f(x, y)$ does not have any limit as $(x, y) \to 0$. This is because if you fix any $r > 0$, no matter how small, $f(x, y)$ takes all values from $-\infty$ to $+\infty$ on the circle $x^2 + y^2 = r^2$. You can see this in the figure on the right below, which shows the level curves yet again, with a circle $x^2 + y^2 = r^2$ superimposed. So for every $\delta > 0$, $f(x, y)$ takes all values from $-\infty$ to $+\infty$ as (x, y) runs over the disk $|(x, y)| < \delta$.



Another way to show that $f(x,y)$ does not have any limit as $(x,y)\to 0$ is to show that $f(x,y)$ does not have a limit as (x,y) approachs $(0,0)$ along some specific curve. This can be done by picking a curve that makes the denominator, $x-y$, tend to zero very quickly. One such curve is $x-y=x^3$ or, equivalently, $y=x-x^3$. Along this curve, for $x\neq 0$,

$$f(x,x-x^3)=\frac{(2x-x+x^3)^2}{x-x+x^3}=\frac{(x+x^3)^2}{x^3}=\frac{(1+x^2)^2}{x}\longrightarrow\begin{cases}+\infty & \text{as } x\to 0 \text{ with } x>0\\ -\infty & \text{as } x\to 0 \text{ with } x<0\end{cases}$$

The figure below shows the level curves, magnified, with the curve $y=x-x^3$ superimposed.



The choice of the power x^3 is not not important. Any power x^p with $p>2$ will have the same effect. If we send (x,y) to $(0,0)$ along the curve $x-y=ax^2$ or, equivalently, $y=x-ax^2$, where a is a constant,

$$\lim_{x\to 0} f(x,x-ax^2)=\lim_{x\to 0}\frac{(2x-x+ax^2)^2}{x-x+ax^2}=\lim_{x\to 0}\frac{(x+ax^2)^2}{ax^2}=\lim_{x\to 0}\frac{(1+ax)^2}{a}=\frac{1}{a}$$

This limit depends on the choice of the constant a. Once again, this proves that $f(x,y)$ does not have a limit as $(x,y)\to 0$.