Taylor Expansions in 2d

In your first year Calculus course you developed a family of formulae for approximating a function $F(t)$ for t near any fixed point t_0. The crudest approximation was just a constant.

$$F(t_0 + \Delta t) \approx F(t_0)$$

The next better approximation included a correction that is linear in Δt.

$$F(t_0 + \Delta t) \approx F(t_0) + F'(t_0)\Delta t$$

The next better approximation included a correction that is quadratic in Δt.

$$F(t_0 + \Delta t) \approx F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2}F''(t_0)\Delta t^2$$

And so on. The approximation that includes all corrections up to order Δt^n is

$$F(t_0 + \Delta t) \approx F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2!}F''(t_0)\Delta t^2 + \tfrac{1}{3!}F^{(3)}(t_0)\Delta t^3 + \cdots + \tfrac{1}{n!}F^{(n)}(t_0)\Delta t^n$$

You may have also found a formula for the error introduced in making this approximation. The error $E_n(\Delta t)$ is defined by

$$F(t_0 + \Delta t) = F(t_0) + F'(t_0)\Delta t + \tfrac{1}{2!}F''(t_0)\Delta t^2 + \cdots + \tfrac{1}{n!}F^{(n)}(t_0)\Delta t^n + E_n(\Delta t)$$

and obeys

$$E_n(\Delta t) = \tfrac{1}{(n+1)!}F^{(n+1)}(t^*)\Delta t^{n+1}$$

for some (unknown) t^* between t_0 and $t_0 + \Delta t$. Even though we do not know what t^* is, we can still learn a lot from this formula for E_n. If we know that $\big|F^{(n+1)}(t)\big| \le M_{n+1}$ for all t between t_0 and $t_0 + \Delta t$, then $\big|E_n(\Delta t)\big| \le \tfrac{M_{n+1}}{(n+1)!}\Delta t^{n+1}$, which tells us that $E_n(\Delta t)$ goes to zero like the $(n+1)^{\rm st}$ power of Δt as Δt tends to zero.

We now generalize all this to functions of more than one variable. To be concrete and to save writing, we'll just look at functions of two variables, but the same strategy works for any number of variables. We'll also assume that all partial derivatives exist and are continuous. Suppose that we wish to approximate $f(x_0 + \Delta x, y_0 + \Delta y)$ for Δx and Δy near zero. The trick is to write

$$f(x_0 + \Delta x, y_0 + \Delta y) = F(1) \text{ with } F(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$$

and think of x_0, y_0, Δx and Δy as constants so that F is a function of the single variable t. Then we can apply our single variable formulae with $t_0 = 0$ and $\Delta t = 1$. To do so we need to compute various derivatives of $F(t)$ at $t = 0$, by applying the chain rule to

$$F(t) = f\big(x(t), y(t)\big) \text{ with } x(t) = x_0 + t\Delta x,\ \ y(t) = y_0 + t\Delta y$$

Since $\frac{d}{dt}x(t) = \Delta x$ and $\frac{d}{dt}y(t) = \Delta y$, the chain rule gives

$$\begin{aligned}
\tfrac{dF}{dt}(t) &= \tfrac{\partial f}{\partial x}\big(x(t),y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f}{\partial y}\big(x(t),y(t)\big)\tfrac{d}{dt}y(t) \\
&= f_x(x_0+t\Delta x, y_0+t\Delta y)\Delta x + f_y(x_0+t\Delta x, y_0+t\Delta y)\Delta y \\
\tfrac{d^2F}{dt^2}(t) &= \Big[\tfrac{\partial f_x}{\partial x}\big(x(t),y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f_x}{\partial y}\big(x(t),y(t)\big)\tfrac{d}{dt}y(t)\Big]\Delta x \\
&\quad + \Big[\tfrac{\partial f_y}{\partial x}\big(x(t),y(t)\big)\tfrac{d}{dt}x(t) + \tfrac{\partial f_y}{\partial y}\big(x(t),y(t)\big)\tfrac{d}{dt}y(t)\Big]\Delta y \\
&= \tfrac{\partial^2 f}{\partial x^2}\Delta x^2 + \tfrac{\partial^2 f}{\partial y\partial x}\Delta y\Delta x + \tfrac{\partial^2 f}{\partial x\partial y}\Delta x\Delta y + \tfrac{\partial^2 f}{\partial y^2}\Delta y^2 \\
&= \tfrac{\partial^2 f}{\partial x^2}\Delta x^2 + 2\tfrac{\partial^2 f}{\partial x\partial y}\Delta x\Delta y + \tfrac{\partial^2 f}{\partial y^2}\Delta y^2
\end{aligned}$$

and so on. It's not hard to prove by induction that, in general, for any natural number k,

$$F^{(k)}(t) = \sum_{\ell=0}^{k} \tbinom{k}{\ell}\tfrac{\partial^k\ f}{\partial x^\ell \partial y^{k-\ell}}(x_0+t\Delta x, y_0+t\Delta y)\Delta x^\ell \Delta y^{k-\ell}$$

where $\binom{k}{\ell} = \frac{k!}{\ell!(k-\ell)!}$ is the standard binomial coefficient. So when $t=0$,

$$\begin{aligned}
F(0) &= f(x_0,y_0) \\
\tfrac{dF}{dt}(0) &= \tfrac{\partial f}{\partial x}(x_0,y_0)\Delta x + \tfrac{\partial f}{\partial y}(x_0,y_0)\Delta y \\
\tfrac{1}{2!}\tfrac{d^2F}{dt^2}(0) &= \tfrac{1}{2}\tfrac{\partial^2 f}{\partial x^2}(x_0,y_0)\Delta x^2 + \tfrac{\partial^2 f}{\partial x\partial y}(x_0,y_0)\Delta x\Delta y + \tfrac{1}{2}\tfrac{\partial^2 f}{\partial y^2}(x_0,y_0)\Delta y^2 \\
\tfrac{1}{k!}\tfrac{d^kF}{dt^k}(0) &= \sum_{\substack{0\le \ell,m\le k\\ \ell+m=k}} \tfrac{1}{\ell!m!}\ \tfrac{\partial^k\ f}{\partial x^\ell \partial y^m}(x_0,y_0)\ \Delta x^\ell \Delta y^m
\end{aligned}$$

Subbing these into

$$f(x_0+\Delta x, y_0+\Delta y) = F(t_0+\Delta t)\big|_{t_0=0,\Delta t=1} = \sum_{k=0}^{n}\tfrac{1}{k!}F^{(k)}(0) + E_n(1)$$

gives

$$\boxed{\begin{aligned}
f(x_0+\Delta x, y_0+\Delta y) &= \sum_{\substack{0\le \ell,m\le n\\ \ell+m\le n}} \tfrac{1}{\ell!m!}\ \tfrac{\partial^{\ell+m}\ f}{\partial x^\ell \partial y^m}(x_0,y_0)\ \Delta x^\ell \Delta y^m + E_n \\
&= f(x_0,y_0) \\
&\quad + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y \\
&\quad + \tfrac{1}{2}\Big[f_{xx}(x_0,y_0)\Delta x^2 + 2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)\Delta y^2\Big] \\
&\quad + E_2
\end{aligned}}$$

If all partial derivatives of f of order $n+1$ are bounded in magnitude by M, then

$$\big|E_n\big| = \tfrac{1}{(n+1)!}\big|F^{(n+1)}(t^*)\big| \le \tfrac{1}{(n+1)!}\sum_{\ell=0}^{n+1}\tbinom{n+1}{\ell}M|\Delta x|^\ell|\Delta y|^{n+1-\ell} = \tfrac{M}{(n+1)!}\big(|\Delta x|+|\Delta y|\big)^{n+1}$$

Example. In this example, we find the second order Taylor expansion of $f(x,y)=\sqrt{1+4x^2+y^2}$ about $(x_0,y_0)=(1,2)$ and use it to compute approximately $f(1.1\,,\,2.05)$. We first compute all partial derivatives up to order 2 at (x_0,y_0).

$$\begin{aligned}
f(x,y) &= \sqrt{1+4x^2+y^2} & f(x_0,y_0) &= 3 \\
f_x(x,y) &= \tfrac{4x}{\sqrt{1+4x^2+y^2}} & f_x(x_0,y_0) &= \tfrac{4}{3} \\
f_y(x,y) &= \tfrac{y}{\sqrt{1+4x^2+y^2}} & f_y(x_0,y_0) &= \tfrac{2}{3} \\
f_{xx}(x,y) &= \tfrac{4}{\sqrt{1+4x^2+y^2}} - \tfrac{16x^2}{[1+4x^2+y^2]^{3/2}} & f_{xx}(x_0,y_0) &= \tfrac{4}{3} - \tfrac{16}{27} = \tfrac{20}{27} \\
f_{xy}(x,y) &= -\tfrac{4xy}{[1+4x^2+y^2]^{3/2}} & f_{xy}(x_0,y_0) &= -\tfrac{8}{27} \\
f_{yy}(x,y) &= \tfrac{1}{\sqrt{1+4x^2+y^2}} - \tfrac{y^2}{[1+4x^2+y^2]^{3/2}} & f_{yy}(x_0,y_0) &= \tfrac{1}{3} - \tfrac{4}{27} = \tfrac{5}{27}
\end{aligned}$$

So the quadratic approximation to f about (x_0,y_0) is

$$\begin{aligned}
f\big(x_0+\Delta x\,,\,y_0+\Delta y\big) &\approx f(x_0,y_0)+f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y \\
&\qquad + \tfrac{1}{2}\Big[f_{xx}(x_0,y_0)\Delta x^2+2f_{xy}(x_0,y_0)\Delta x\Delta y+f_{yy}(x_0,y_0)\Delta y^2\Big] \\
&= 3+\tfrac{4}{3}\Delta x+\tfrac{2}{3}\Delta y+\tfrac{10}{27}\Delta x^2-\tfrac{8}{27}\Delta x\Delta y+\tfrac{5}{54}\Delta y^2
\end{aligned}$$

In particular, with $\Delta x=0.1$ and $\Delta y=0.05$,

$$f(1.1\,,\,2.05)\approx 3+\tfrac{4}{3}(0.1)+\tfrac{2}{3}(0.05)+\tfrac{10}{27}(0.01)-\tfrac{8}{27}(0.005)+\tfrac{5}{54}(0.0025)=3.1691$$

The actual value, to four decimal places, is 3.1690.

Example. In this example, we find the third order Taylor expansion of $f(x,y)=e^{2x}\sin(3y)$ about $(x_0,y_0)=(0,0)$ in two different ways. The first way uses the canned formula. We first compute all partial derivatives up to order 3 at (x_0,y_0).

$$\begin{aligned}
f(x,y) &= e^{2x}\sin(3y) & f(x_0,y_0) &= 0 \\
f_x(x,y) &= 2e^{2x}\sin(3y) & f_x(x_0,y_0) &= 0 \\
f_y(x,y) &= 3e^{2x}\cos(3y) & f_y(x_0,y_0) &= 3 \\
f_{xx}(x,y) &= 4e^{2x}\sin(3y) & f_{xx}(x_0,y_0) &= 0 \\
f_{xy}(x,y) &= 6e^{2x}\cos(3y) & f_{xy}(x_0,y_0) &= 6 \\
f_{yy}(x,y) &= -9e^{2x}\sin(3y) & f_{yy}(x_0,y_0) &= 0 \\
f_{xxx}(x,y) &= 8e^{2x}\sin(3y) & f_{xx}(x_0,y_0) &= 0 \\
f_{xxy}(x,y) &= 12e^{2x}\cos(3y) & f_{xxy}(x_0,y_0) &= 12 \\
f_{xyy}(x,y) &= -18e^{2x}\sin(3y) & f_{xyy}(x_0,y_0) &= 0 \\
f_{yyy}(x,y) &= -27e^{2x}\cos(3y) & f_{yyy}(x_0,y_0) &= -27
\end{aligned}$$

So the Taylor expansion, about $(0,0)$ to order three is

$$\begin{aligned}
f(x,y) &= \sum_{\substack{0\le \ell,m\le 3\\ \ell+m\le 3}} \tfrac{1}{\ell!m!}\ \tfrac{\partial^{\ell+m} f}{\partial x^\ell \partial y^m}(0,0)\ x^\ell y^m + E_3(x,y)\\
&= \tfrac{1}{0!1!}3y + \tfrac{1}{1!1!}6xy + \tfrac{1}{2!1!}12x^2y - \tfrac{1}{0!3!}27y^3 + E_3(x,y)\\
&= 3y + 6xy + 6x^2y - \tfrac{9}{2}y^3 + E_3(x,y)
\end{aligned}$$

A second way to get the same result exploits the single variable Taylor expansions

$$\begin{aligned}
e^x &= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots\\
\sin y &= y - \tfrac{1}{3!}y^3 + \cdots
\end{aligned}$$

Replacing x by $2x$ in the first and y by $3y$ in the second and multiplying the two together, keeping track only of terms of degree at most three, gives

$$\begin{aligned}
f(x,y) &= e^{2x}\sin(3y)\\
&= \Big[1 + (2x) + \tfrac{1}{2!}(2x)^2 + \tfrac{1}{3!}(2x)^3 + \cdots\Big]\Big[(3y) - \tfrac{1}{3!}(3y)^3 + \cdots\Big]\\
&= \Big[1 + 2x + 2x^2 + \tfrac{4}{3}x^3 + \cdots\Big]\Big[3y - \tfrac{9}{2}y^3 + \cdots\Big]\\
&= 3y - \tfrac{9}{2}y^3 + 6xy + 6x^2y + \cdots
\end{aligned}$$

just as in the first computation.

Problem 1 Derive the Taylor expansion, to order 2 in powers of Δx, Δy and Δz, of

$$f\big(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z\big)$$

Answer: In general, the expansion to order N is

$$f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) = \sum_{\substack{0\le \ell,m,n\le 2\\ \ell+m+n\le N}} \tfrac{1}{\ell!m!n!}\ \tfrac{\partial^{\ell+m+n} f}{\partial x^\ell \partial y^m \partial z^n}(x_0,y_0,z_0)\ (\Delta x)^\ell (\Delta y)^m (\Delta z)^n + E_n$$

Problem 2 Find the second order Taylor expansion of $f(x,y) = \frac{1}{1+x^2+y^2}$ about $(x_0,y_0) = (1,-1)$.

Answer: $f(1+\Delta x, -1+\Delta y) = \frac{1}{3} - \frac{2}{9}\Delta x + \frac{2}{9}\Delta y + \frac{1}{27}\Delta x^2 - \frac{8}{27}\Delta x\Delta y + \frac{1}{27}\Delta y^2 + \cdots$

Problem 3 Find the second order Taylor expansion of $f(x,y,z) = \frac{1}{1+x^2+y^2+z^2}$ about $(x_0,y_0,z_0) = (0,0,0)$.

Answer: $f(x,y,z) = 1 - x^2 - y^2 - z^2 + \cdots$

Problem 4 Find the second order Taylor expansion of $f(x,y) = e^{x+2y}$ about $(x_0, y_0) = (0,0)$. Bound the error of this approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

Answer: $f(x,y) = 1 + x + 2y + \frac{1}{2}x^2 + 2xy + 2y^2 + E_2(x,y)$ with $\big|E_2(x,y)\big| \le \frac{(0.3)^3}{3!}e^{0.3}$ when $|x| \le 0.1$ and $|y| \le 0.1$. (Of course there are many other correct bounds on $\big|E_2(x,y)\big|$.)

Problem 5 The function $z = f(x,y)$ solves the equation $x + 2y + z + e^{2z} = 1$. Find the Taylor polynomial of degree two for $f(x,y)$ in powers of x and y.

Answer: $f(x,y) = -\frac{1}{3}x - \frac{2}{3}y - \frac{2}{27}x^2 - \frac{8}{27}xy - \frac{8}{27}y^2 + \cdots$

Problem 6 Suppose that $f(x,y)$ is a polynomial of degree n. Prove that for all real numbers x_0, y_0, Δx and Δy,

$$f(x_0 + \Delta x, y_0 + \Delta y) = \sum_{\substack{0 \le \ell, m \le n \\ \ell + m \le n}} \tfrac{1}{\ell! m!} \tfrac{\partial^{\ell+m} f}{\partial x^\ell \partial y^m}(x_0, y_0)\ (\Delta x)^\ell (\Delta y)^m$$

Note that this formula is exact. There is no error term E_n.