Properties of Vector Operations

Addition and Scalar Multiplication

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Dot Product

The dot product is defined by

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \,,\ \vec{b} = \langle b_1, b_2, b_3 \rangle$$
$$\implies \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and obeys

0. $\vec{a}, \vec{b}$ are vectors and $\vec{a} \cdot \vec{b}$ is a number
1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$
6. $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$
7. $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \perp \vec{b}$

In property 6, θ is the angle between $\vec{a}$ and $\vec{b}$.

Cross Product

The cross product is defined by

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \,,\ \vec{b} = \langle b_1, b_2, b_3 \rangle$$
$$\implies \vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

and obeys

0. $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are all vectors in three dimensions
1. $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$
2. $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$
3. $\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{\jmath}$
4. $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\ \hat{n}$
5. $\vec{a} \times \vec{b} = 0 \iff \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$
6. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
7. $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$
8. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
9. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
10. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$

In properties 2 and 4, θ is the angle between $\vec{a}$ and $\vec{b}$. In property 4, $|\hat{n}| = 1$, $\hat{n} \perp \vec{a}, \vec{b}$ and $(\vec{a}, \vec{b}, \hat{n})$ obey the right hand rule.

WARNING: Take particular care with properties 6 and 10. They are counterintuitive and cause huge numbers of errors. In particular,

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$
$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

for most $\vec{a}$, $\vec{b}$ and $\vec{c}$. For example

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{k} \times \hat{\imath} = -\hat{\jmath}$$
$$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \hat{\jmath} = \vec{0}$$