Proofs that $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$

Proof One:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (a_1, a_2, a_3) \cdot (b_2 c_3 - b_3 c_2, b_3 c_1 - b_1 c_3, b_1 c_2 - b_2 c_1)$$
$$= a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1$$
$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \cdot (c_1, c_2, c_3)$$
$$= a_2 b_3 c_1 - a_3 b_2 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 + a_1 b_2 c_3 - a_2 b_1 c_3$$

Proof Two:

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (a_1, a_2, a_3) \cdot \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \cdot (c_1, c_2, c_3)$$

$$= c_1 \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - c_2 \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + c_3 \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$= \det \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Exchanging two rows in a determinant changes the sign of the determinant. Moving the top row of a 3×3 determinant to the bottom row requires two exchanges of rows. So the two 3×3 determinants are equal.