The Astroid

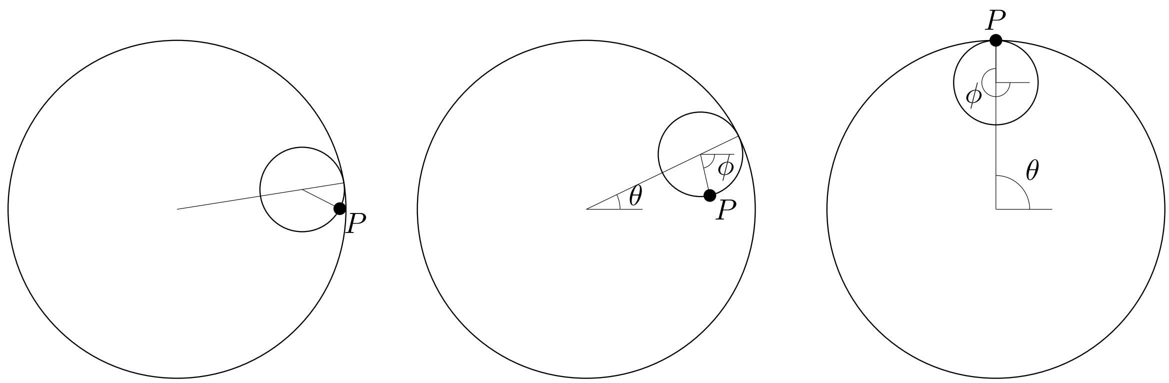
Imagine a ball of radius $a/4$ rolling around the inside of a circle of radius a. We shall now find the equation of the curve traced by a point P painted on the inner circle. You can find a java applet demonstrating this curve on the web in the bottom half of the page

http://www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/implicit.html

There is also a link to this applet on our course home page

http://www.math.ubc.ca/~feldman/m227/

Define the angles θ and ϕ as in the middle figure below.



That is,

- the vector from the centre of the circle of radius a to the centre of the ball of radius $a/4$ is $\frac{3}{4}a\big(\cos\theta, \sin\theta\big)$ and
- the vector from the centre of the ball of radius $a/4$ to the point P is $\frac{1}{4}a\big(\cos\phi, -\sin\phi\big)$

As θ runs from 0 to $\frac{\pi}{2}$, the point of contact between the two circles travels through one quarter of the circumference of the circle of radius a, which is a distance $\frac{1}{4}(2\pi a)$, which, in turn, is exactly the circumference of the inner circle. Hence if $\phi = 0$ for $\theta = 0$ (i.e. if P starts on the x–axis), then for $\theta = \frac{\pi}{2}$, P is back in contact with the big circle at the north pole of both the inner and outer circles. That is, $\phi = \frac{3\pi}{2}$ when $\theta = \frac{\pi}{2}$. So $\phi = 3\theta$ and P has coordinates

$$\frac{3}{4}a\big(\cos\theta, \sin\theta\big) + \frac{1}{4}a\big(\cos\phi, -\sin\phi\big) = \frac{a}{4}\big(3\cos\theta + \cos 3\theta, 3\sin\theta - \sin 3\theta\big)$$

As

$$\begin{aligned}
\cos 3\theta &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta \\
&= \cos\theta[\cos^2\theta - \sin^2\theta] - 2\sin^2\theta\cos\theta \\
&= \cos\theta[\cos^2\theta - 3\sin^2\theta] \\
\sin 3\theta &= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta \\
&= \sin\theta[\cos^2\theta - \sin^2\theta] + 2\sin\theta\cos^2\theta \\
&= \sin\theta[3\cos^2\theta - \sin^2\theta]
\end{aligned}$$

we have

$$3\cos\theta + \cos 3\theta = \cos\theta[3+\cos^2\theta - 3\sin^2\theta] = \cos\theta[3+\cos^2\theta - 3(1-\cos^2\theta)] = 4\cos^3\theta$$
$$3\sin\theta - \sin 3\theta = \sin\theta[3-3\cos^2\theta + \sin^2\theta] = \sin\theta[3-3(1-\sin^2\theta)+\sin^2\theta] = 4\sin^3\theta$$

and the coordinates of P simplify to

$$x(\theta) = a\cos^3\theta \qquad y(\theta) = a\sin^3\theta$$

As $x^{2/3} + y^{2/3} = a^{2/3}\cos^2\theta + a^{2/3}\sin^2\theta$, the path traced by P obeys the equation

$$\boxed{x^{2/3} + y^{2/3} = a^{2/3}}$$

There remains the danger that there could exist points (x, y) obeying the equation $x^{2/3}+y^{2/3} = a^{2/3}$ that are not of the form $x = a\cos^3\theta,\ y = a\sin^3\theta$ for any θ. That is, there is a danger that the parametrized curve $x = a\cos^3\theta,\ y = a\sin^3\theta$ covers only a portion of $x^{2/3} + y^{2/3} = a^{2/3}$. We now show that the parametrized curve $x = a\cos^3\theta,\ y = a\sin^3\theta$ in fact covers all of $x^{2/3} + y^{2/3} = a^{2/3}$ as θ runs from 0 to 2π. First, observe that $x^{2/3} = \left(\sqrt[3]{x}\right)^2 \geq 0$ and $y^{2/3} = \left(\sqrt[3]{y}\right)^2 \geq 0$. Hence, if (x, y) obeys $x^{2/3} + y^{2/3} = a^{2/3}$, then necessarily $0 \leq x^{2/3} \leq a^{2/3}$ and so $-a \leq x \leq a$. As θ runs from 0 to 2π, $a\cos^3\theta$ takes all values between $-a$ and a and hence takes all possible values of x. For each $x \in [-a, a]$, y takes two values, namely $\pm[a^{2/3} - x^{2/3}]^{3/2}$. If $x = a\cos^3\theta_0 = a\cos^3(2\pi - \theta_0)$, the two corresponding values of y are precisely $a\sin^3\theta_0$ and $-a\sin^3\theta_0 = a\sin^3(2\pi - \theta_0)$.