Buoyancy

In these notes, we use the divergence theorem to show that when you immerse a body in a fluid the net effect of fluid pressure acting on the surface of the body is a vertical force (called the buoyant force) whose magnitude equals the weight of fluid displaced by the body. This is known as Archimedes' principle. We shall also show that the buoyant force acts through the "centre of buoyancy" which is the centre of mass of the fluid displaced by the body. The design of self righting boats exploits the fact that the centre of buoyancy and the centre of gravity, where gravity acts, need not be the same.

We start by computing the total force due to the pressure of the fluid pushing on the body. Recall that pressure

- is the force per unit surface area that the fluid exerts on the body
- acts perpendicularly to the surface
- pushes on the body

Thus the force due to pressure that acts on an infinitesimal piece of body surface at $\mathbf{r} = (x, y, z)$ with surface area dS and outward normal $\hat{\mathbf{n}}$ is $-p(\mathbf{r})\,\hat{\mathbf{n}}dS$. The minus sign is there because pressure is directed into the body. If the body fills the volume V and has surface ∂V, then the total force on the body due to fluid pressure is

$$\mathbf{B} = -\iint_{\partial V} p(\mathbf{r})\,\hat{\mathbf{n}}\,dS$$

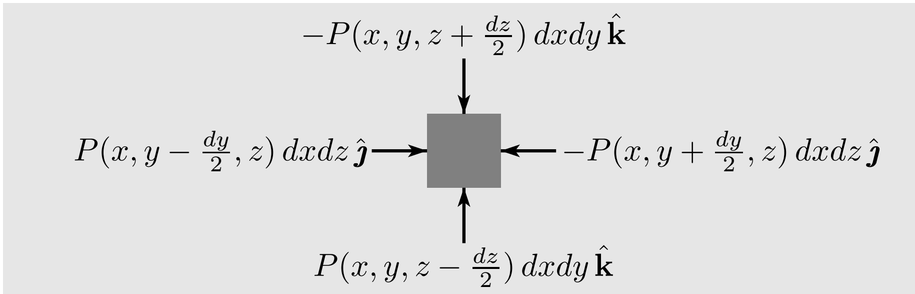
We now wish to apply a variant of the divergence theorem to rewrite $\mathbf{B} = -\iiint_V \boldsymbol{\nabla} p\ dV$. But there is a problem with this: $p(\mathbf{r})$ is the fluid pressure at $\mathbf{r}$ and is only defined where there is fluid. In particular, there is no fluid inside the body, so $p(\mathbf{r})$ is not defined for $\mathbf{r}$ in the interior of V. So we pretend that we remove the body from the fluid and we call $P(\mathbf{r})$ the fluid pressure at $\mathbf{r}$ when there is no body in the fluid. We also make the assumption that at any point $\mathbf{r}$ outside of the body, the pressure at $\mathbf{r}$ does not depend on whether the body is in the fluid or not. In other words, we assume that

$$p(\mathbf{r}) = \begin{cases} P(\mathbf{r}) & \text{if } \mathbf{r} \text{ is not in } V \\ \text{not defined} & \text{if } \mathbf{r} \text{ is in the } V \end{cases}$$

This assumption is only an approximation to reality, but, in practice, it is a very good approximation. So

$$\mathbf{B} = -\iint_{\partial V} p(\mathbf{r})\,\hat{\mathbf{n}}\,dS = -\iint_{\partial V} P(\mathbf{r})\,\hat{\mathbf{n}}\,dS = -\iiint_V \boldsymbol{\nabla} P(\mathbf{r})\,dV \tag{1}$$

Our next job is to compute $\boldsymbol{\nabla} P$. Concentrate on an infinitesimal cube of fluid whose edges are parallel to the coordinate axes. Call the lengths of the edges dx, dy and dz and the position of the centre of the cube (x, y, z). The forces applied to the various faces of the cube by the pressure of fluid outside the cube are illustrated in the figure



The total force due to the pressure acting on the cube is

$$\begin{aligned}
&- P\big(x + \tfrac{dx}{2}, y, z\big)\, dydz\, \hat{\boldsymbol{\imath}} + P\big(x - \tfrac{dx}{2}, y, z\big)\, dydz\, \hat{\boldsymbol{\imath}} \\
&- P\big(x, y + \tfrac{dy}{2}, z\big)\, dxdz\, \hat{\boldsymbol{\jmath}} + P\big(x, y - \tfrac{dy}{2}, z\big)\, dxdz\, \hat{\boldsymbol{\jmath}} \\
&- P\big(x, y, z + \tfrac{dz}{2}\big)\, dxdy\, \hat{\mathbf{k}} + P\big(x, y, z - \tfrac{dz}{2}\big)\, dxdy\, \hat{\mathbf{k}}
\end{aligned}$$

Rewriting

$$-P\big(x + \tfrac{dx}{2}, y, z\big)\, dydz\, \hat{\boldsymbol{\imath}} + P\big(x - \tfrac{dx}{2}, y, z\big)\, dydz\, \hat{\boldsymbol{\imath}} = -\frac{P(x+\frac{dx}{2},y,z)-P(x+\frac{dx}{2},y,z)}{dx}\hat{\boldsymbol{\imath}}\; dxdydz$$

and rewriting the remaining four terms in a similar manner, we see that the total force due to the pressure acting on the cube is

$$-\Big\{\tfrac{\partial P}{\partial x}(x, y, z)\, \hat{\boldsymbol{\imath}} + \tfrac{\partial P}{\partial y}(x, y, z)\, \hat{\boldsymbol{\jmath}} + \tfrac{\partial P}{\partial z}(x, y, z)\, \hat{\mathbf{k}}\Big\} dxdydz = -\boldsymbol{\nabla} P(x, y, z)\, dxdydz$$

We shall assume that the only other force acting on the cube is gravity and that the fluid is stationary (or at least not accelerating) so that the total force acting on the cube is zero. If the fluid has density ρ_f, then the cube has mass $\rho_f\, dxdydz$ so that the force of gravity is $-g\rho_f\, dxdydz\, \hat{\mathbf{k}}$. The vanishing of the total force now tells us that

$$-\boldsymbol{\nabla} P(\mathbf{r})\, dxdydz - g\rho_f\, dxdydz\, \hat{\mathbf{k}} = 0 \implies \boldsymbol{\nabla} P(\mathbf{r}) = -g\rho_f\, \hat{\mathbf{k}}$$

Subbing this into (1) gives

$$\mathbf{B} = g\, \hat{\mathbf{k}} \iiint_V \rho_f\, dV = gM_f\, \hat{\mathbf{k}}$$

where $M_f = \iiint_V \rho_f\, dV$ is the mass of the fluid displaced by the body. Thus the buoyant force acts straight up and has magnitude equal to gM_f, which is also the magnitude of the force of gravity acting on the fluid displaced by the body. In other words, it is the weight of the displaced fluid. This is exactly Archimedes' principle.

We next consider the rotational motion of our submerged body. The physical law that determines the rotational motion of a rigid body about a point $\mathbf{r}_0$ is analogous to the familiar Newton's law, $m\frac{d\mathbf{v}}{dt} = \mathbf{F}$, that determines the translational motion of the body. For the rotational law of motion,

- the mass m is replaced by a physical quantity, characteristic of the body, called the moment of inertia, and
- the ordinary velocity $\mathbf{v}$ is replaced by the angular velocity, which is a vector whose length is the rate of rotation (i.e. angle rotated per unit time) and whose direction is parallel to the axis of rotation (with the sign determined by a right hand rule) and
- the force $\mathbf{F}$ is replaced by a vector called the torque about $\mathbf{r}_0$. A force $\mathbf{F}$ applied at $\mathbf{r} = (x, y, z)$ produces the torque $(\mathbf{r} - \mathbf{r}_0) \times \mathbf{F}$ about $\mathbf{r}_0$.

This is derived in the notes "Torque" and is all that we need to know about rotational motion of rigid bodies in these notes.

Fix any point $\mathbf{r}_0$. The total torque about $\mathbf{r}_0$ produced by pressure acting on the surface of the submerged body is

$$\mathbf{T} = \iint_{\partial V} (\mathbf{r} - \mathbf{r}_0) \times \big(- p(\mathbf{r})\hat{\mathbf{n}}\big)\, dS = \iint_{\partial V} \hat{\mathbf{n}} \times \big(P(\mathbf{r})\,(\mathbf{r} - \mathbf{r}_0)\big)\, dS$$

Recall that in these integrals $\mathbf{r} = (x, y, z)$ is the position of the infinitesmal piece dS of the surface S. By the cross product variant of the divergence theorem,

$$\begin{aligned}\mathbf{T} &= \iiint_V \boldsymbol{\nabla} \times \big(P(\mathbf{r})\,(\mathbf{r} - \mathbf{r}_0)\big)\, dV = \iiint_V \big\{\boldsymbol{\nabla} P(\mathbf{r}) \times (\mathbf{r} - \mathbf{r}_0) + P(\mathbf{r})\boldsymbol{\nabla} \times (\mathbf{r} - \mathbf{r}_0)\big\}\, dV \\ &= \iiint_V \boldsymbol{\nabla} P(\mathbf{r}) \times (\mathbf{r} - \mathbf{r}_0)\, dV\end{aligned}$$

since $\boldsymbol{\nabla} \times \mathbf{r}_0 = 0$, because $\mathbf{r}_0$ is a constant, and

$$\boldsymbol{\nabla} \times \mathbf{r} = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = 0$$

Subbing in $\boldsymbol{\nabla} P(\mathbf{r}) = -g\rho_f\hat{\mathbf{k}}$

$$\begin{aligned}\mathbf{T} &= -\iiint_V g\rho_f\hat{\mathbf{k}} \times (\mathbf{r} - \mathbf{r}_0)\, dV & &= -g\hat{\mathbf{k}} \times \iiint_V \rho_f(\mathbf{r} - \mathbf{r}_0)\, dV \\ &= -g\hat{\mathbf{k}} \times \Big\{ \iiint_V \mathbf{r}\rho_f\, dV - \mathbf{r}_0 \iiint_V \rho_f\, dV \Big\} & &= -g\Big\{ \iiint_V \rho_f\, dV \Big\}\hat{\mathbf{k}} \times \Big\{ \frac{\iiint_V \mathbf{r}\rho_f\, dV}{\iiint_V \rho_f\, dV} - \mathbf{r}_0 \Big\} \\ &= -\mathbf{B} \times \Big\{ \frac{\iiint_V \mathbf{r}\rho_f\, dV}{\iiint_V \rho_f\, dV} - \mathbf{r}_0 \Big\} & &= \Big\{ \frac{\iiint_V \mathbf{r}\rho_f\, dV}{\iiint_V \rho_f\, dV} - \mathbf{r}_0 \Big\} \times \mathbf{B}\end{aligned}$$

So the torque generated at $\mathbf{r}_0$ by pressure over the entire surface is the same as a force $\mathbf{B}$ all applied at the single point

$$\mathbf{C_B} = \frac{\iiint_V \mathbf{r}\rho_f\, dV}{\iiint_V \rho_f\, dV}$$

This point is called the centre of buoyancy. It is the centre of mass of the displaced fluid. The moral of the above discussion is that the buoyant force, $\mathbf{B}$, on a rigid body

- acts straight upward,
- has magnitude equal to the weight of the displaced fluid and
- acts at the centre of buoyancy, which is the centre of mass of the displaced fluid.

Similarly, the gravitational force, $\mathbf{G}$,

- acts straight down,
- has magnitude equal to the weight $gM_b = g\iiint_V \mathbf{r}\rho_b\, dV$ (where ρ_b is the density of the body) of the body and
- acts at the centre of mass, $\mathbf{C_G} = \dfrac{\iiint_V \mathbf{r}\rho_b\, dV}{\iiint_V \rho_b\, dV}$, of the body.

Because the mass distribution of the body need not be the same as the mass distribution of the displaced fluid, buoyancy and gravity may act at two different points. This is exploited in the design of self–righting boats. These boats are constructed with a heavy, often lead, keel. As a result, the centre of gravity is lower in the boat than the center of buoyancy, which is at the geometric centre of the boat. As the figure below illustrates, a right side up configuration of such a boat is stable, while an upside down configuration is unstable. The boat rotates so as to keep the centre of gravity straight below the centre of buoyancy.