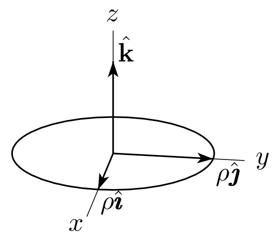
# Parametrizing Circles

These notes discuss a simple strategy for parametrizing circles in three dimensions. We start with the circle in the $xy$–plane that has radius $\rho$ and is centred on the origin. This is easy to parametrize:



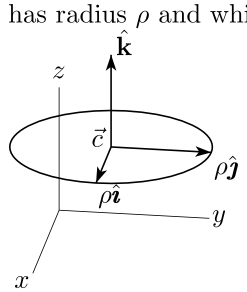

$$\vec r(t) = \rho\cos t\,\hat{\imath} + \rho\sin t\,\hat{\jmath} \qquad 0 \le t \le 2\pi$$

Note that we can check that $\vec r(t)$ lies on the desired circle by checking, firstly, that $\vec r(t)$ lies in the correct plane (in this case, the $xy$–plane) and, secondly, that the distance from $\vec r(t)$ to the centre of the circle is $\rho$:

$$\left|\vec r(t) - \vec 0\right| = \left|\rho\cos t\,\hat{\imath} + \rho\sin t\,\hat{\jmath}\right| = \sqrt{(\rho\cos t)^2 + (\rho\sin t)^2} = \rho$$

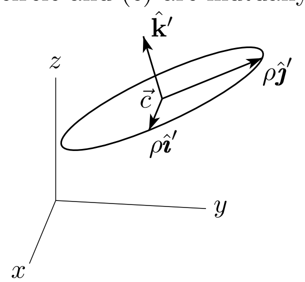
since $\sin^2 t + \cos^2 t = 1$.

Now let's move the circle so that its centre is at some general point $\vec c$. To parametrize this new circle, which still has radius $\rho$ and which is still parallel to the $xy$–plane, we just translate by $\vec c$:




$$\vec r(t) = \vec c + \rho\cos t\,\hat{\imath} + \rho\sin t\,\hat{\jmath} \qquad 0 \le t \le 2\pi$$

Finally, let's consider a circle in general position. The secret to parametrizing a general circle is to replace $\hat{\imath}$ and $\hat{\jmath}$ by two new vectors $\hat{\imath}'$ and $\hat{\jmath}'$ which (a) are unit vectors, (b) are parallel to the plane of the desired circle and (c) are mutually perpendicular.




$$\vec r(t) = \vec c + \rho\cos t\,\hat{\imath}' + \rho\sin t\,\hat{\jmath}' \qquad 0 \le t \le 2\pi$$

To check that this is correct, observe that

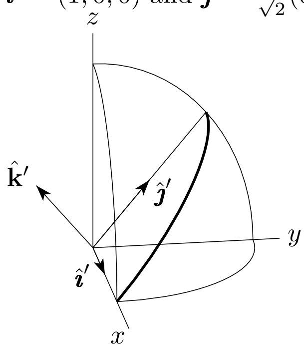
- $\vec r(t) - \vec c$ is parallel to the plane of the desired circle because $\vec r(t) - \vec c = \rho\cos t\,\hat{\imath}' + \rho\sin t\,\hat{\jmath}'$ and both $\hat{\imath}'$ and $\hat{\jmath}'$ are parallel to the plane of the desired circle
- $\vec r(t) - \vec c$ is of length $\rho$ for all $t$ because

$$\begin{aligned}
|\vec r(t) - \vec c\,|^2 &= (\vec r(t) - \vec c\,)\cdot(\vec r(t) - \vec c\,) \\
&= (\rho\cos t\,\hat{\imath}' + \rho\sin t\,\hat{\jmath}')\cdot(\rho\cos t\,\hat{\imath}' + \rho\sin t\,\hat{\jmath}') \\
&= \rho^2\cos^2 t\ \hat{\imath}'\cdot\hat{\imath}' + \rho^2\sin^2 t\ \hat{\jmath}'\cdot\hat{\jmath}' + 2\rho\cos t\sin t\ \hat{\imath}'\cdot\hat{\jmath}' \\
&= \rho^2(\cos^2 t + \sin^2 t) = \rho^2
\end{aligned}$$

since $\hat{\imath}'\cdot\hat{\imath}' = \hat{\jmath}'\cdot\hat{\jmath}' = 1$ ($\hat{\imath}'$ and $\hat{\jmath}'$ are both unit vectors) and $\hat{\imath}'\cdot\hat{\jmath}' = 0$ ($\hat{\imath}'$ and $\hat{\jmath}'$ are perpendicular).

To find such a parametrization in practice, we need to find the centre $\vec{c}$ of the circle, the radius $\rho$ of the circle and two mutually perpendicular unit vectors, $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, in the plane of the circle. It is often easy to find at least one point $\vec{p}$ on the circle. Then we can take $\hat{\boldsymbol{\imath}}' = \frac{\vec{p}-\vec{c}}{|\vec{p}-\vec{c}|}$. It is also often easy to find a unit vector, $\hat{\mathbf{k}}'$, that is normal to the plane of the circle. Then we can choose $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}'$.

**Example 1** Let $C$ be the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = y$. The intersection of any plane with any sphere is a circle. The plane in question passes through the centre of the sphere, so $C$ has the same centre and same radius as the sphere. So $C$ has radius 2 and centre $(0,0,0)$. The point $(2,0,0)$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$ and so is on $C$. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(0,0,0)$ of the circle towards $(2,0,0)$. Namely $\hat{\boldsymbol{\imath}}' = (1,0,0)$. Since the plane of the circle is $z - y = 0$, the vector $\vec{\nabla}(z-y) = (0,-1,1)$ is perpendicular to the plane of $C$. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{2}}(0,-1,1)$. Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{2}}(0,-1,1) \times (1,0,0) = \frac{1}{\sqrt{2}}(0,1,1)$. Subbing in $\vec{c} = (0,0,0)$, $\rho = 2$, $\hat{\boldsymbol{\imath}}' = (1,0,0)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0,1,1)$ gives



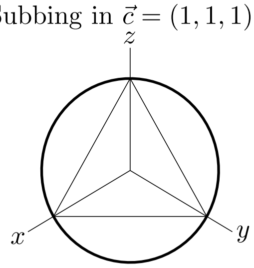

$$\vec{r}(t) = 2\cos t\,(1,0,0) + 2\sin t\,\tfrac{1}{\sqrt{2}}(0,1,1) = 2\big(\cos t, \tfrac{\sin t}{\sqrt{2}}, \tfrac{\sin t}{\sqrt{2}}\big) \qquad 0 \le t \le 2\pi$$

To check this, note that $x = 2\cos t$, $y = \sqrt{2}\sin t$, $z = \sqrt{2}\sin t$ satisfies both $x^2 + y^2 + z^2 = 4$ and $z = y$.

**Example 2** Let $C$ be the circle that passes through the three points $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$. All three points obey $x + y + z = 3$. So the circle lies in the plane $x + y + z = 3$. We guess, by symmetry, or by looking at the figure below, that the centre of the circle is at the centre of mass of the three points, which is $\frac{1}{3}[(3,0,0) + (0,3,0) + (0,0,3)] = (1,1,1)$. We can check this by checking that $(1,1,1)$ is equidistant from the three points:

$$\begin{aligned}
\big|(3,0,0) - (1,1,1)\big| &= \big|(2,-1,-1)\big| = \sqrt{6}\\
\big|(0,3,0) - (1,1,1)\big| &= \big|(-1,2,-1)\big| = \sqrt{6}\\
\big|(0,0,3) - (1,1,1)\big| &= \big|(-1,-1,2)\big| = \sqrt{6}
\end{aligned}$$

This tells us both that $(1,1,1)$ is indeed the centre and that the radius of $C$ is $\sqrt{6}$. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(1,1,1)$ of the circle towards $(3,0,0)$. Namely $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2,-1,-1)$. Since the plane of the circle is $x+y+z = 3$, the vector $\vec{\nabla}(x+y+z) = (1,1,1)$ is perpendicular to the plane of $C$. So we may take $\hat{\mathbf{k}}' = \frac{1}{\sqrt{3}}(1,1,1)$. Then $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{18}}(1,1,1) \times (2,-1,-1) = \frac{1}{\sqrt{18}}(0,3,-3) = \frac{1}{\sqrt{2}}(0,1,-1)$. Subbing in $\vec{c} = (1,1,1)$, $\rho = \sqrt{6}$, $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(2,-1,-1)$ and $\hat{\boldsymbol{\jmath}}' = \frac{1}{\sqrt{2}}(0,1,-1)$ gives




$$\begin{aligned}
\vec{r}(t) &= (1,1,1) + \sqrt{6}\cos t\,\tfrac{1}{\sqrt{6}}(2,-1,-1) + \sqrt{6}\sin t\,\tfrac{1}{\sqrt{2}}(0,1,-1)\\
&= \big(1 + 2\cos t, 1 - \cos t + \sqrt{3}\sin t, 1 - \cos t - \sqrt{3}\sin t\big) \qquad 0 \le t \le 2\pi
\end{aligned}$$

To check this, note that $\vec{r}(0) = (3,0,0)$, $\vec{r}\big(\frac{2\pi}{3}\big) = (0,3,0)$ and $\vec{r}\big(\frac{4\pi}{3}\big) = (0,0,3)$ since $\cos\frac{2\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.