Circulation around a small circle

Let C_ε be the circle which

- is centered on $\vec{\mathbf{r}}_0$
- has radius ε
- lies in the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$
- is oriented in the standard way with respect to $\hat{\mathbf{n}}$. Imagine standing on the circle with your feet on the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$, with the vector from your feet to your head in the same direction as $\hat{\mathbf{n}}$ and with your left arm point towards $\vec{\mathbf{r}}_0$. Then your are facing in the positive direction for C_ε.

We shall show that

$$\oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \pi\varepsilon^2 \vec{\nabla} \times v(\vec{\mathbf{r}}_0) \cdot \hat{\mathbf{n}} + O(\varepsilon^3)$$

To do so, pick any three vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, $\hat{\mathbf{k}}'$ such that

- $\hat{\mathbf{k}}' = \hat{\mathbf{n}}$
- $\hat{\boldsymbol{\imath}}' \perp \hat{\mathbf{n}}$, $\hat{\boldsymbol{\jmath}}' \perp \hat{\mathbf{n}}$
- $\hat{\boldsymbol{\imath}}' \times \hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}'$

Then

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'$$

is a parametrization of C_ε. Note in particular that, for all t, $\vec{\mathbf{r}}(t)$ lies in the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$ and $\|\vec{\mathbf{r}}(t) - \vec{\mathbf{r}}_0\| = \varepsilon$. So

$$\oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \int_0^{2\pi} \vec{v}\big(\vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'\big) \cdot \big(-\varepsilon \sin t\, \hat{\boldsymbol{\imath}}' + \varepsilon \cos t\, \hat{\boldsymbol{\jmath}}'\big) dt$$
$$= \varepsilon \int_0^{2\pi} \Big[-\sin t\, \hat{\boldsymbol{\imath}}' \cdot \vec{v}\big(\vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'\big) + \cos t\, \hat{\boldsymbol{\jmath}}' \cdot \vec{v}\big(\vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'\big)\Big]\, dt$$

Denote $f(\vec{\mathbf{r}}) = \hat{\boldsymbol{\imath}}' \cdot \vec{v}(\vec{\mathbf{r}})$. Now Taylor expand $F(\varepsilon) = f\big(\vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'\big)$ about $\varepsilon = 0$.

$$\begin{aligned} F(\varepsilon) &= F(0) + F'(0)\varepsilon + O(\varepsilon^2) \\ &= f\big(\vec{\mathbf{r}}_0\big) + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' \cdot \vec{\nabla} f\big(\vec{\mathbf{r}}_0\big) + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}' \cdot \vec{\nabla} f\big(\vec{\mathbf{r}}_0\big) + O(\varepsilon^2) \\ &= \hat{\boldsymbol{\imath}}' \cdot \vec{v}\big(\vec{\mathbf{r}}_0\big) + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' \cdot \vec{\nabla}(\hat{\boldsymbol{\imath}}' \cdot \vec{v})\big(\vec{\mathbf{r}}_0\big) + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}' \cdot \vec{\nabla}(\hat{\boldsymbol{\imath}}' \cdot \vec{v})\big(\vec{\mathbf{r}}_0\big) + O(\varepsilon^2) \end{aligned}$$

Similarly, if $G(\varepsilon) = \hat{\boldsymbol{\jmath}}' \cdot \vec{v}\big(\vec{\mathbf{r}}_0 + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}'\big)$,

$$G(\varepsilon) = \hat{\boldsymbol{\jmath}}' \cdot \vec{v}\big(\vec{\mathbf{r}}_0\big) + \varepsilon \cos t\, \hat{\boldsymbol{\imath}}' \cdot \vec{\nabla}(\hat{\boldsymbol{\jmath}}' \cdot \vec{v})\big(\vec{\mathbf{r}}_0\big) + \varepsilon \sin t\, \hat{\boldsymbol{\jmath}}' \cdot \vec{\nabla}(\hat{\boldsymbol{\jmath}}' \cdot \vec{v})\big(\vec{\mathbf{r}}_0\big) + O(\varepsilon^2)$$

Hence, the integrand

$$\begin{aligned}
&-\sin t\ \hat{\imath}'\cdot\vec v\big(\vec{\mathbf r}_0+\varepsilon\cos t\ \hat{\imath}'+\varepsilon\sin t\ \hat{\jmath}'\big)+\cos t\ \hat{\jmath}'\cdot\vec v\big(\vec{\mathbf r}_0+\varepsilon\cos t\ \hat{\imath}'+\varepsilon\sin t\ \hat{\jmath}'\big)\\
=\ &-\sin t\ \hat{\imath}'\cdot\vec v\big(\vec{\mathbf r}_0\big)-\varepsilon\sin t\cos t\ \hat{\imath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)-\varepsilon\sin^2 t\ \hat{\jmath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)\\
&+\cos t\ \hat{\jmath}'\cdot\vec v\big(\vec{\mathbf r}_0\big)+\ \varepsilon\cos^2 t\ \hat{\imath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)\ +\varepsilon\sin t\cos t\ \hat{\jmath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)+O(\varepsilon^2)
\end{aligned}$$

Since

$$\int_0^{2\pi}\sin t\ dt=\int_0^{2\pi}\cos t\ dt\int_0^{2\pi}\sin t\cos t\ dt=0\qquad\int_0^{2\pi}\sin^2 t\ dt=\int_0^{2\pi}\cos^2 t\ dt=\pi$$

we have

$$\oint_{C_\varepsilon}\vec v(\vec{\mathbf r})\cdot d\vec{\mathbf r}=\pi\varepsilon^2\Big[-\hat{\jmath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)+\hat{\imath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)\Big]+O(\varepsilon^3)$$

Sub in

$$\hat{\imath}'=-\hat{\mathbf k}'\times\hat{\jmath}'\qquad\hat{\jmath}'=\hat{\mathbf k}'\times\hat{\imath}'\qquad\vec\nabla=\sum_{n=1}^3\hat{\imath}_n\tfrac{\partial}{\partial x_n}$$

where I have renamed the standard basis for $\mathbb{R}^3$ from $\hat{\imath},\hat{\jmath},\hat{\mathbf k}$ to $\hat{\imath}_1,\hat{\imath}_2\hat{\imath}_3$ and the standard coordinates on $\mathbb{R}^3$ from x,y,z to x_1,x_2,x_3. Then

$$\begin{aligned}
-\hat{\jmath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)+\hat{\imath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)&=\sum_{n=1}^3-\hat{\jmath}'\cdot\hat{\imath}_n\tfrac{\partial\hat{\imath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)+\sum_{n=1}^3\hat{\imath}'\cdot\hat{\imath}_n\tfrac{\partial\hat{\jmath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)\\
&=\sum_{n=1}^3-(\hat{\mathbf k}'\times\hat{\imath}')\cdot\hat{\imath}_n\tfrac{\partial\hat{\imath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)+\sum_{n=1}^3-(\hat{\mathbf k}'\times\hat{\jmath}')\cdot\hat{\imath}_n\tfrac{\partial\hat{\jmath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)\\
&=\sum_{n=1}^3-\hat{\mathbf k}'\cdot(\hat{\imath}'\times\hat{\imath}_n)\tfrac{\partial\hat{\imath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)+\sum_{n=1}^3-\hat{\mathbf k}'\cdot(\hat{\jmath}'\times\hat{\imath}_n)\tfrac{\partial\hat{\jmath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)\\
&=\sum_{n=1}^3\hat{\mathbf k}'\cdot(\hat{\imath}_n\times\hat{\imath}')\tfrac{\partial\hat{\imath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)+\sum_{n=1}^3\hat{\mathbf k}'\cdot(\hat{\imath}_n\times\hat{\jmath}')\tfrac{\partial\hat{\jmath}'\cdot\vec v}{\partial x_n}\big(\vec{\mathbf r}_0\big)
\end{aligned}$$

Since $\hat{\mathbf k}'\perp\hat{\imath}_n\times\hat{\mathbf k}')$, $\hat{\mathbf k}'\cdot(\hat{\imath}_n\times\hat{\mathbf k}')=0$ and

$$-\hat{\jmath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)+\hat{\imath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)=\sum_{n=1}^3\hat{\mathbf k}'\cdot\hat{\imath}_n\times\tfrac{\partial}{\partial x_n}\big[\hat{\imath}'(\hat{\imath}'\cdot\vec v)+\hat{\jmath}'(\hat{\jmath}'\cdot\vec v)+\hat{\mathbf k}'(\hat{\mathbf k}'\cdot\vec v)\big]\big(\vec{\mathbf r}_0\big)$$

For any orthonormal basis, $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{\mathbf k}'$ and any vector $\vec v$,

$$\hat{\imath}'(\hat{\imath}'\cdot\vec v)+\hat{\jmath}'(\hat{\jmath}'\cdot\vec v)+\hat{\mathbf k}'(\hat{\mathbf k}'\cdot\vec v)=\vec v$$

so

$$-\hat{\jmath}'\cdot\vec\nabla(\hat{\imath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)+\hat{\imath}'\cdot\vec\nabla(\hat{\jmath}'\cdot\vec v)\big(\vec{\mathbf r}_0\big)=\sum_{n=1}^3\hat{\mathbf k}'\cdot\hat{\imath}_n\times\tfrac{\partial\vec v}{\partial x_n}(\vec{\mathbf r}_0)=\hat{\mathbf k}'\cdot\vec\nabla\times\vec v(\vec{\mathbf r}_0)$$

and

$$\oint_{C_\varepsilon}\vec v(\vec{\mathbf r})\cdot d\vec{\mathbf r}=\pi\varepsilon^2\vec\nabla\times v(\vec{\mathbf r}_0)\cdot\hat{\mathbf n}+O(\varepsilon^3)$$