Complex Numbers and Exponentials

Definition and Basic Operations

A complex number is nothing more than a point in the xy–plane. The first component, x, of the complex number (x, y) is called its real part and the second component, y, is called its imaginary part, even though there is nothing imaginary about it. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) is defined by

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1)\,(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

respectively. We'll get an effective memory aid for the definition of multiplication shortly. It is conventional to use the notation $x + iy$ (or in electrical engineering country $x + jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, the sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$\begin{aligned}z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2)\\ z_1z_2 &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\end{aligned}$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{aligned}z_1 + z_2 &= z_2 + z_1 & z_1z_2 &= z_2z_1\\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2z_3) &= (z_1z_2)z_3\\ 0 + z_1 &= z_1 & 1z_1 &= z_1\\ z_1(z_2 + z_3) &= z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3\end{aligned}$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$. The inverse, z^{-1} or $\frac{1}{z}$, of any complex number $z = x + iy$, other than 0, is defined by $\frac{1}{z}z = 1$. We shall see below that it is given by the formula $\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$. The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

To remember how to multiply complex numbers, you just have to supplement the familiar rules of the real number system with $i^2 = -1$. For example, if $z = 1 + 2i$ and $w = 3 + 4i$, then

$$\begin{aligned}z + w &= (1 + 2i) + (3 + 4i) = 4 + 6i\\ zw &= (1 + 2i)(3 + 4i) \quad = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i\end{aligned}$$

Other Operations

The complex conjugate of z is denoted $\bar{z}$ and is defined to be $\bar{z} = x - iy$. That is, to take the complex conjugate, one replaces every i by $-i$. Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a positive real number. In fact, it is the square of the distance from $x+iy$ (recall that this is the point (x,y) in the xy–plane) to 0 (which is the point $(0,0)$). The distance from $z=x+iy$ to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z. It is given by

$$|z| \;=\; \sqrt{x^2+y^2} \;=\; \sqrt{z\bar{z}}$$

Since $z_1z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$,

$$\begin{aligned}
|z_1z_2| &= \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2-2x_1x_2y_1y_2+y_1^2y_2^2+x_1^2y_2^2+2x_1y_2x_2y_1+x_2^2y_1^2} \\
&= \sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} \\
&= |z_1||z_2|
\end{aligned}$$

for all complex numbers z_1, z_2.

Since $|z|^2 = z\bar{z}$, we have $z\big(\frac{\bar{z}}{|z|^2}\big) = 1$ for all complex numbers $z \neq 0$. This says that the multiplicative inverse, $\frac{1}{z}$, of any nonzero complex number $z=x+iy$ is

$$z^{-1} = \tfrac{\bar{z}}{|z|^2} = \tfrac{x-iy}{x^2+y^2} = \tfrac{x}{x^2+y^2} - \tfrac{y}{x^2+y^2}i$$

This is the formula for $\frac{1}{z}$ given above. It is easy to divide a complex number by a real number. For example

$$\tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

In general, there is a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3-4i$ is the complex conjugate of $3+4i$. Since $(3+4i)(3-4i) = 9-12i+12i+16 = 25$

$$\tfrac{1+2i}{3+4i} = \tfrac{1+2i}{3+4i}\tfrac{3-4i}{3-4i} = \tfrac{(1+2i)(3-4i)}{25} = \tfrac{11+2i}{25} = \tfrac{11}{25} + \tfrac{2}{25}i$$

The notations $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ stand for the real and imaginary parts of the complex number z, respectively. If $z=x+iy$ (with x and y real) they are defined by

$$\mathrm{Re}\, z = x \qquad \mathrm{Im}\, z = y$$

Note that both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are real numbers. Just subbing in $\bar{z} = x-iy$ gives

$$\mathrm{Re}\, z = \tfrac{1}{2}(z+\bar{z}) \qquad \mathrm{Im}\, z = \tfrac{1}{2i}(z-\bar{z})$$

The Complex Exponential

Definition and Basic Properties. For any complex number $z=x+iy$ the exponential e^z, is defined by

$$e^{x+iy} \;=\; e^x\cos y + ie^x \sin y$$

In particular, $e^{iy} \;=\; \cos y + i\sin y$. This definition is not as mysterious as it looks. We could also define e^{iy} by the subbing x by iy in the Taylor series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{iy} = 1 + iy + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^5}{5!} + \tfrac{(iy)^6}{6!} + \cdots$$

The even terms in this expansion are

$$1 + \tfrac{(iy)^2}{2!} + \tfrac{(iy)^4}{4!} + \tfrac{(iy)^6}{6!} + \cdots = 1 - \tfrac{y^2}{2!} + \tfrac{y^4}{4!} - \tfrac{y^6}{6!} + \cdots = \cos y$$

and the odd terms in this expansion are

$$iy + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^5}{5!} + \cdots = i\Big(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} + \cdots\Big) = i\sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}\left\{(\cos y_1\cos y_2 - \sin y_1\sin y_2) + i(\cos y_1\sin y_2 + \cos y_2\sin y_1)\right\} \\ &= e^{x_1+x_2}\left\{\cos(y_1+y_2) + i\sin(y_1+y_2)\right\} \\ &= e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{z_1+z_2} \end{aligned}$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number $c = \alpha + i\beta$ and real number t

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned} \tfrac{d}{dt}e^{ct} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)] \\ &= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\ &= ce^{ct} \end{aligned}$$

is also the familiar one.

Relationship with $\sin$ **and** $\cos$**.** When θ is a real number

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta = \overline{e^{i\theta}} \end{aligned}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \operatorname{Re} e^{i\theta} \\ \sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im} e^{i\theta} \end{aligned}$$

These formulae make it easy derive trig identities. For example

$$\begin{aligned} \cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi}) \\ &= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}) \\ &= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}) \\ &= \tfrac{1}{2}\big(\cos(\theta+\phi) + \cos(\theta-\phi)\big) \end{aligned}$$

and, using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

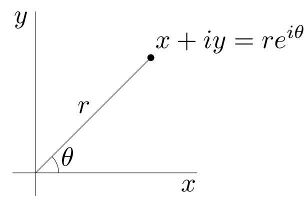
$$\begin{aligned} \sin^3\theta &= -\tfrac{1}{8i}\big(e^{i\theta} - e^{-i\theta}\big)^3 \\ &= -\tfrac{1}{8i}\big(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\big) \\ &= \tfrac{3}{4}\tfrac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big) - \tfrac{1}{4}\tfrac{1}{2i}\big(e^{i3\theta} - e^{-i3\theta}\big) \\ &= \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta) \end{aligned}$$

and

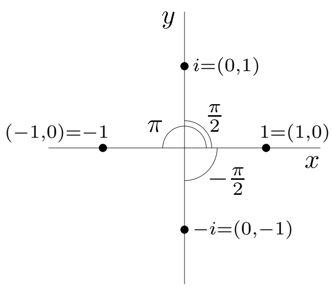
$$\begin{aligned}\cos(2\theta) &= \operatorname{Re} e^{i2\theta} = \operatorname{Re}\left(e^{i\theta}\right)^2 \\ &= \operatorname{Re}\big(\cos\theta + i\sin\theta\big)^2 \\ &= \operatorname{Re}\big(\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\big) \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Polar Coordinates. Let $z = x + iy$ be any complex number. Writing (x, y) in polar coordinates in the usual way gives $x = r\cos\theta$, $y = r\sin\theta$ and

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



In particular



$$\begin{aligned} 1 &= e^{i0} = e^{2\pi i} = e^{2k\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots \\ -1 &= e^{i\pi} = e^{3\pi i} = e^{(1+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots \\ i &= e^{i\pi/2} = e^{\frac{5}{2}\pi i} = e^{(\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots \\ -i &= e^{-i\pi/2} = e^{\frac{3}{2}\pi i} = e^{(-\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots \end{aligned}$$

The polar coordinate $\theta = \tan^{-1}\frac{y}{x}$ associated with the complex number $z = x + iy$ is also called the argument of z.

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The $n^{\rm th}$ roots of unity are, by definition, all solutions z of

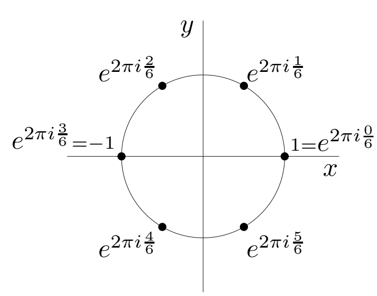
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy–plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The $n^{\rm th}$ roots of unity are all complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct $n^{\rm th}$ roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi i\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The n distinct nth roots of unity are

$$1\ ,\ e^{2\pi i\frac{1}{n}}\ ,\ e^{2\pi i\frac{2}{n}}\ ,\ e^{2\pi i\frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i\frac{n-1}{n}}$$



Exploiting Complex Exponentials in Calculus Computations

Example 1

$$\int e^x \cos x\ dx = \tfrac{1}{2}\int e^x\big[e^{ix}+e^{-ix}\big]\ dx = \tfrac{1}{2}\int \big[e^{(1+i)x}+e^{(1-i)x}\big]\ dx$$
$$= \tfrac{1}{2}\big[\tfrac{1}{1+i}e^{(1+i)x}+\tfrac{1}{1-i}e^{(1-i)x}\big]+C$$

This form of the indefinite integral looks a little wierd because of the i's. But it is correct and it is purely real, despite the i's, because $\frac{1}{1-i}e^{(1-i)x}$ is the complex conjugate of $\frac{1}{1+i}e^{(1+i)x}$. We can convert the indefinite integral into a more familar form just by subbing back in $e^{\pm ix}=\cos x\pm i\sin x$, $\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{2}$ and $\frac{1}{1-i}=\overline{\frac{1}{1+i}}=\frac{1+i}{2}$.

$$\begin{aligned}\int e^x\cos x\ dx &= \tfrac{1}{2}e^x\big[\tfrac{1}{1+i}e^{ix}+\tfrac{1}{1-i}e^{-ix}\big]+C\\ &= \tfrac{1}{2}e^x\big[\tfrac{1-i}{2}(\cos x+i\sin x)+\tfrac{1+i}{2}(\cos x-i\sin x)\big]+C\\ &= \tfrac{1}{2}e^x\cos x+\tfrac{1}{2}e^x\sin x+C\end{aligned}$$

Example 2 Using $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$,

$$\begin{aligned}\int \cos^4 x\ dx &= \tfrac{1}{2^4}\int\big[e^{ix}+e^{-ix}\big]^4\ dx = \tfrac{1}{2^4}\int\big[e^{4ix}+4e^{2ix}+6+4e^{-2ix}+e^{-4ix}\big]\ dx\\ &= \tfrac{1}{2^4}\big[\tfrac{1}{4i}e^{4ix}+\tfrac{4}{2i}e^{2ix}+6x+\tfrac{4}{-2i}e^{-2ix}+\tfrac{1}{-4i}e^{-4ix}\big]+C\\ &= \tfrac{1}{2^4}\big[\tfrac{1}{2}\tfrac{1}{2i}(e^{4ix}-e^{-4ix})+\tfrac{4}{2i}(e^{2ix}-e^{-2ix})+6x\big]+C\\ &= \tfrac{1}{2^4}\big[\tfrac{1}{2}\sin 4x+4\sin 2x+6x\big]+C\\ &= \tfrac{1}{32}\sin 4x+\tfrac{1}{4}\sin 2x+\tfrac{3}{8}x+C\end{aligned}$$

Example 3 We shall now guess a solution to the differential equation

$$y''+2y'+3y=\cos t \tag{1}$$

Equations like this arise, for example, in the study of the RLC circuit. We shall simplify the computation by exploiting that $\cos t=\operatorname{Re}e^{it}$. First, we shall guess a function $Y(t)$ obeying

$$Y''+2Y'+3Y=e^{it} \tag{2}$$

Then, taking complex conjugates,

$$\bar{Y}''+2\bar{Y}'+3\bar{Y}=e^{-it} \tag{$\bar{2}$}$$

and, adding $\frac{1}{2}(2)$ and $\frac{1}{2}(\bar{2})$ together will give

$$(\operatorname{Re}Y)''+2(\operatorname{Re}Y)'+3(\operatorname{Re}Y)=\operatorname{Re}e^{it}=\cos t$$

which shows that $\operatorname{Re}Y(t)$ is a solution to (1). Let's try $Y(t)=Ae^{it}$. This is a solution of (2) if and only if

$$\begin{aligned}& \tfrac{d^2}{dt^2}\big(Ae^{it}\big)+2\tfrac{d}{dt}\big(Ae^{it}\big)+3Ae^{it}=e^{it}\\ \iff\quad & (2+2i)Ae^{it}=e^{it}\\ \iff\quad & A=\tfrac{1}{2+2i}\end{aligned}$$

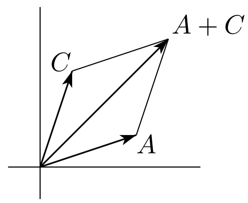
So we have found a solution to (2) and $\operatorname{Re}\frac{e^{it}}{2+2i}$ is a solution to (1). To simplify this, write $2+2i$ in polar coordinates. So

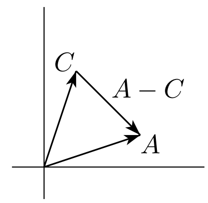
$$2+2i=2\sqrt{2}e^{i\frac{\pi}{4}}\Rightarrow\tfrac{e^{it}}{2+2i}=\tfrac{e^{it}}{2\sqrt{2}e^{i\frac{\pi}{4}}}=\tfrac{1}{2\sqrt{2}}e^{i(t-\frac{\pi}{4})}\Rightarrow\operatorname{Re}\tfrac{e^{it}}{2+2i}=\tfrac{1}{2\sqrt{2}}\cos(t-\tfrac{\pi}{4})$$

Sketching Complex Numbers as Vectors

Algebraic expressions involving complex numbers may be evaluated geometrically by exploiting the following two observations.

- (Addition and subtraction) A complex number is nothing more than a point in the xy–plane. So we may identify the complex number $A = a + ib$ with the vector whose tail is at the origin and whose head is at the point (a, b). Similarly, we may identify the complex number $C = c + id$ with the vector whose tail is at the origin and whose head is at the point (c, d). Those two vectors form two sides of a parallelogram. The vector for the sum $A+C = (a+c)+i(b+d)$ is that from the origin to the diagonally opposite corner of the parallelogram. The vector for the difference $A - C = (a - c) + i(b - d)$ has its tail at C and its head at A.





- (Multiplication and Division) To multiply or divide two complex numbers, write them in their polar coordinate forms $A = re^{i\theta}$, $C = \rho e^{i\varphi}$. So r and ρ are the lengths of A and C, respectively, and θ and φ are the angles from the positive x–axis to A and C, respectively. Then $AC = r\rho e^{i(\theta+\varphi)}$. This vector has length equal to the product of the lengths of A and C. The angle from the positive x–axis to AC is the sum of the angles θ and φ. And $\frac{A}{C} = \frac{r}{\rho}e^{i(\theta-\varphi)}$. This vector has length equal to the ratio of the lengths of A and C. The angle from the positive x–axis to AC is the difference of the angles θ and φ.