A Compendium of Curve Formulae

In the following $\mathbf{r}(t) = \big(x(t)\,,\, y(t)\,,\, z(t)\big)$ is a parametrization of a curve. The vectors $\hat{\mathbf{T}}(t)$, $\hat{\mathbf{N}}(t)$, and $\hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t)$ are the unit tangent, normal and binormal vectors, respectively, at $\mathbf{r}(t)$. The tangent vector points in the direction of travel (i.e. direction of increasing t) and the normal vector points toward the centre of curvature. The arc length from time 0 to time t is denoted $s(t)$. Then

- the velocity $\quad \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t) = \frac{ds}{dt}(t)\,\hat{\mathbf{T}}(t)$
- the acceleration $\quad \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}(t) = \frac{d^2s}{dt^2}(t)\,\hat{\mathbf{T}}(t) + \kappa(t)\big(\frac{ds}{dt}(t)\big)^2\hat{\mathbf{N}}(t)$
- the speed $\quad \frac{ds}{dt}(t) = |\mathbf{v}(t)| = \big|\frac{d\mathbf{r}}{dt}(t)\big|$
- the arc length $\quad s(t) = \int_0^t \frac{ds}{dt}(\tau)\ d\tau = \int_0^1 \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\ d\tau$
- the curvature $\quad \kappa(t) = \dfrac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{\big(\frac{ds}{dt}(t)\big)^3}$
- the radius of curvature $\quad \rho(t) = \frac{1}{\kappa(t)}$
- the centre of curvature is $\quad \mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t)$
- the torsion $\quad \tau(t) = \dfrac{\big(\mathbf{v}(t) \times \mathbf{a}(t)\big) \cdot \frac{d\mathbf{a}}{dt}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2}$
- the binormal $\quad \hat{\mathbf{B}}(t) = \hat{\mathbf{T}}(t) \times \hat{\mathbf{N}}(t) = \dfrac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|}$

Under arclength parametrization (i.e. if $t = s$) we have $\hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds}(s)$ and the Frenet-Serret formulae

$$\begin{aligned}
\tfrac{d\hat{\mathbf{T}}}{ds}(s) &= \quad \kappa(s)\ \hat{\mathbf{N}}(s) \\
\tfrac{d\hat{\mathbf{N}}}{ds}(s) &= \quad \tau(s)\ \hat{\mathbf{B}}(s) - \kappa(s)\ \hat{\mathbf{T}}(s) \\
\tfrac{d\hat{\mathbf{B}}}{ds}(s) &= -\tau(s)\ \hat{\mathbf{N}}(s)
\end{aligned}$$

When the curve lies in the x–y plane

$$\kappa(t) = \frac{\big|\frac{dx}{dt}(t)\frac{d^2y}{dt^2}(t) - \frac{dy}{dt}(t)\frac{d^2x}{dt^2}(t)\big|}{\Big[\big(\frac{dx}{dt}(t)\big)^2 + \big(\frac{dy}{dt}(t)\big)^2\Big]^{3/2}}$$

When the curve lies in the x–y plane and the parameter is x (so that y is given as a function $y(x)$ of x)

$$\kappa(x) = \frac{\big|\frac{d^2y}{dx^2}(x)\big|}{\Big[1 + \big(\frac{dy}{dx}(x)\big)^2\Big]^{3/2}}$$