A Continuous Bijection with Discontinuous Inverse

If you take Math 320 next year, you will learn that if a map

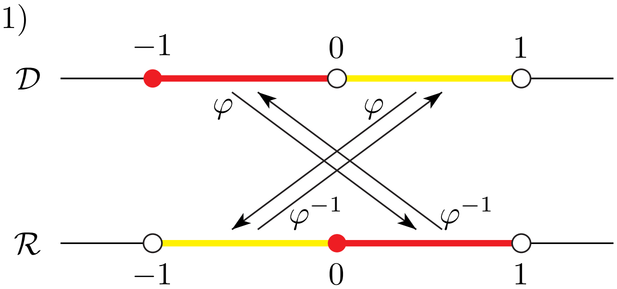
- is continuous and
- is bijective (meaning that it is one–to–one and onto) and
- has a compact domain (a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded)

then

- its inverse map is also continuous.

In theses notes, we will see what can go wrong if the domain is not compact. We will construct a map $\varphi : \mathcal{D} \to \mathcal{R}$ which is continuous, one–to–one and onto (i.e. bijective) and whose inverse map is not continuous. The domain $\mathcal{D}$ and range $\mathcal{R}$ are

$$\mathcal{D} = [-1, 0) \cup (0, 1)$$
$$\mathcal{R} = (-1, 1)$$



The map

$$y = \varphi(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0 \\ x-1 & \text{if } 0 < x < 1 \end{cases}$$

So

$$\varphi\big([-1,0)\big) = [0,1) \qquad \varphi\big((0,1)\big) = (-1,0) \qquad \varphi(\mathcal{D}) = \mathcal{R}$$

The map φ is continuous — because 0 is not in the domain of φ, we need worry about its continuity there. The inverse map of φ is

$$x = \varphi^{-1}(y) = \begin{cases} y-1 & \text{if } 0 \le y < 1 \\ y+1 & \text{if } -1 < y < 0 \end{cases}$$

So

$$\varphi^{-1}\big([0,1)\big) = [-1,0) \qquad \varphi^{-1}\big((-1,0)\big) = (0,1) \qquad \varphi^{-1}(\mathcal{R}) = \mathcal{D}$$

Now 0 is in the domain of φ^{-1} and φ^{-1} is *not* continuous there.