Divergence Theorem and Variations

Theorem. *If V is a solid with surface ∂V*

$$\iint_{\partial V} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\ dV$$

$$\iint_{\partial V} f\hat{\mathbf{n}}\,dS = \iiint_V \boldsymbol{\nabla} f\ dV$$

$$\iint_{\partial V} \hat{\mathbf{n}}\times\mathbf{F}\,dS = \iiint_V \boldsymbol{\nabla}\times\mathbf{F}\ dV$$

where $\hat{\mathbf{n}}$ is the outward unit normal of ∂V.

Memory Aid. All three formulae can be combined into

$$\iint_{\partial V} \hat{\mathbf{n}} * \tilde{F}\,dS = \iiint_V \boldsymbol{\nabla} * \tilde{F}\ dV$$

where $*$ can be either $\cdot$, $\times$ or nothing. When $* = \cdot$ or $* = \times$, then $\tilde{F} = \mathbf{F}$. When $*$ is nothing, $\tilde{F} = f$.

Proof: The first formula is the divergence theorem and was proven in class.

To prove the second formula, assuming the first, apply the first with $\mathbf{F} = f\mathbf{a}$, where $\mathbf{a}$ is any constant vector.

$$\begin{aligned}\iint_{\partial V} f\mathbf{a}\cdot\hat{\mathbf{n}}\,dS &= \iiint_V \boldsymbol{\nabla}\cdot(f\mathbf{a})\ dV\\ &= \iiint_V \big[(\boldsymbol{\nabla} f)\cdot\mathbf{a} + f\boldsymbol{\nabla}\cdot\mathbf{a}\big]\ dV\\ &= \iiint_V (\boldsymbol{\nabla} f)\cdot\mathbf{a}\ dV\end{aligned}$$

To get the second line, we used vector identity # 8. To get the third line, we just used that $\mathbf{a}$ is a constant, so that it is annihilated by all derivatives. Since $\mathbf{a}$ is a constant, we can factor it out of both integrals, so

$$\mathbf{a}\cdot\iint_{\partial V} f\hat{\mathbf{n}}\,dS = \mathbf{a}\cdot\iiint_V \boldsymbol{\nabla} f\ dV$$

$$\Longrightarrow \mathbf{a}\cdot\left\{\iint_{\partial V} f\hat{\mathbf{n}}\,dS - \iiint_V \boldsymbol{\nabla} f\ dV\right\} = 0$$

In particular, choosing $\mathbf{a} = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V} f \hat{\mathbf{n}}\, dS - \iiint_V \boldsymbol{\nabla} f\ dV$ are zero. So

$$\iint\limits_{\partial V} f \hat{\mathbf{n}}\, dS - \iiint_V \boldsymbol{\nabla} f\ dV = 0$$

which is what we wanted show.

To prove the third formula, assuming the first, apply the first with $\mathbf{F}$ replaced by $\mathbf{F} \times \mathbf{a}$, where $\mathbf{a}$ is any constant vector.

$$\begin{aligned} \iint\limits_{\partial V} \mathbf{F} \times \mathbf{a} \cdot \hat{\mathbf{n}}\ dS &= \iiint_V \boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{a})\ dV \\ &= \iiint_V \big[(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{a} - \mathbf{F} \cdot \boldsymbol{\nabla} \times \mathbf{a}\big]\ dV \\ &= \iiint_V (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{a}\ dV \end{aligned}$$

To get the second line, we used vector identity # 9. To get the third line, we just used that $\mathbf{a}$ is a constant, so that it is annihilated by all derivatives. For all vectors

$$\mathbf{F} \times \mathbf{a} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{F} \times \mathbf{a} = \hat{\mathbf{n}} \times \mathbf{F} \cdot \mathbf{a}$$

so

$$\begin{aligned} \mathbf{a} \cdot \iint\limits_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\ dS &= \mathbf{a} \cdot \iiint_V \boldsymbol{\nabla} \times \mathbf{F}\ dV \\ \Longrightarrow \mathbf{a} \cdot \Big\{ \iint\limits_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\ dS &- \iiint_V \boldsymbol{\nabla} \times \mathbf{F}\ dV \Big\} = 0 \end{aligned}$$

In particular, choosing $\mathbf{a} = \hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\ dS - \iiint_V \boldsymbol{\nabla} \times \mathbf{F}\ dV$ are zero. So

$$\iint\limits_{\partial V} \hat{\mathbf{n}} \times \mathbf{F}\ dS - \iiint_V \boldsymbol{\nabla} \times \mathbf{F}\ dV = 0$$

which is what we wanted show. ■