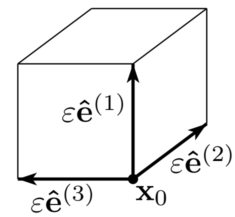
# The Physical Significance of div and curl

Consider a (possibly compressible) fluid with velocity field $\mathbf{v}(\mathbf{x},t)$. Pick any time $t_0$ and a really tiny piece of the fluid, which at time $t_0$ is a cube with corners at

$$\left\{\ \mathbf{x}_0 + n_1\varepsilon\hat{\mathbf{e}}^{(1)} + n_2\varepsilon\hat{\mathbf{e}}^{(2)} + n_3\varepsilon\hat{\mathbf{e}}^{(3)}\ \middle|\ n_1,n_2,n_3\in\{0,1\}\ \right\}$$




Here $\varepsilon$ is the length of each edge of the cube and is assumed to be really small. The vectors $\hat{\mathbf{e}}^{(1)}$, $\hat{\mathbf{e}}^{(2)}$ and $\hat{\mathbf{e}}^{(3)}$ are three mutually perpendicular unit vectors that give the orientation of the cube. The vectors from the corner $\mathbf{x}_0$ to its three nearest neighbour corners are $\varepsilon\hat{\mathbf{e}}^{(1)}$, $\varepsilon\hat{\mathbf{e}}^{(2)}$ and $\varepsilon\hat{\mathbf{e}}^{(3)}$. As time progresses, the hunk of fluid moves. In particular, the corners move. Let us denote by $\varepsilon\mathbf{b}^{(1)}(t)$ the vector, at time $t$, joining the $n_1=n_2=n_3=0$ corner to the $n_1=1,\ n_2=n_3=0$ corner. Define $\varepsilon\mathbf{b}^{(2)}(t)$ and $\varepsilon\mathbf{b}^{(3)}(t)$ similarly. For times very close to $t_0$ we can think of our hunk of fluid as being essentially a parallelepiped with edges $\varepsilon\mathbf{b}^{(k)}(t)$. By concentrating on the edges $\varepsilon\mathbf{b}^{(k)}(t)$ of the hunk of fluid, rather than the corners, like $\mathbf{x}_0$, we are ignoring the translations of the hunk of fluid. We want, instead, to determine how the size and orientation of the parallelepiped changes as $t$ increases.

At time $t_0$, $\mathbf{b}^{(k)}=\hat{\mathbf{e}}^{(k)}$. The velocities of the corners of the hunk of fluid at time $t_0$ are

$$\mathbf{v}\big(\mathbf{x}_0 + n_1\varepsilon\hat{\mathbf{e}}^{(1)} + n_2\varepsilon\hat{\mathbf{e}}^{(2)} + n_3\varepsilon\hat{\mathbf{e}}^{(3)}, t_0\big)$$

In particular, at time $t_0$, the tail of $\varepsilon\mathbf{b}^{(k)}$ has velocity $\mathbf{v}(\mathbf{x}_0,t_0)$ and the head of $\varepsilon\mathbf{b}^{(k)}$ has velocity $\mathbf{v}(\mathbf{x}_0+\varepsilon\hat{\mathbf{e}}^{(k)},t_0)$. Consequently,

$$\varepsilon\tfrac{d\,\mathbf{b}^{(k)}}{d\,t}(t_0) = \mathbf{v}\big(\mathbf{x}_0+\varepsilon\hat{\mathbf{e}}^{(k)},t_0\big) - \mathbf{v}\big(\mathbf{x}_0,t_0\big) = \sum_{j=1}^{3}\varepsilon\tfrac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0,t_0\big)\hat{\mathbf{e}}_j^{(k)} + O(\varepsilon^2)$$

and

$$\tfrac{d\,\mathbf{b}^{(k)}}{d\,t}(t_0) = \sum_{j=1}^{3}\tfrac{\partial\mathbf{v}}{\partial x_j}\big(\mathbf{x}_0,t_0\big)\hat{\mathbf{e}}_j^{(k)} + O(\varepsilon)$$

The notation $O(\varepsilon^n)$ represents a function that is bounded by a constant times $\varepsilon^n$ for all sufficiently small $\varepsilon$. The notation $\hat{\mathbf{e}}_j^{(k)}$ just refers to the $j^{\rm th}$ component of the vector $\hat{\mathbf{e}}^{(k)}$.

Define the $3\times3$ matrix $\mathcal{V}$ by

$$\mathcal{V}_{i,j} = \tfrac{\partial v_i}{\partial x_j}\big(\mathbf{x}_0,t_0\big) \tag{M}$$

Then

$$\tfrac{d\,\mathbf{b}^{(k)}}{d\,t}(t_0) = \mathcal{V}\mathbf{b}^{(k)}(t_0) + O(\varepsilon)$$

We study the behaviour of $\mathbf{b}^{(k)}(t)$ for small $\varepsilon$ and $t$ close to $t_0$, by studying the behaviour of the solutions to the initial value problems

$$\tfrac{d\,\mathbf{b}^{(k)}}{d\,t}(t) = \mathcal{V}\mathbf{b}^{(k)}(t) \qquad \mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)} \tag{IVP}$$

To warm up, we first look at two two–dimensional examples. In both examples, the velocity field $\mathbf{v}(x, y)$ is linear in $(x, y)$. Consequently, in these examples, $\mathbf{v}\big(\mathbf{x}_0+\varepsilon\hat{\mathbf{e}}^{(k)}, t_0\big)-\mathbf{v}\big(\mathbf{x}_0, t_0\big)$ is exactly $\sum_{j=1}^{3}\varepsilon\frac{\partial \mathbf{v}}{\partial x_j}\big(\mathbf{x}_0, t_0\big)\hat{\mathbf{e}}_j^{(k)}$ and the solution to (IVP) coincides with the exact $\mathbf{b}^{(k)}(t)$. Following each example, we discuss a broad class of $\mathcal{V}$'s that generate behaviour similar to that of the example.

**Example 1:** $\mathbf{v}(x, y) = 2x\hat{\boldsymbol{\imath}} + 3y\hat{\boldsymbol{\jmath}}$.

In this example

$$\mathcal{V} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

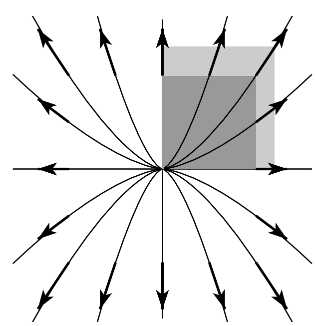
The solution to

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad \text{or equivalently} \quad \begin{array}{ll} b_1'(t) = 2b_1(t) & b_1(0) = \beta_1 \\ b_2'(t) = 3b_2(t) & b_2(0) = \beta_2 \end{array}$$

is

$$\begin{array}{l} b_1(t) = e^{2t}\beta_1 \\ b_2(t) = e^{3t}\beta_2 \end{array} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}\mathbf{b}(0)$$

If one chooses $\hat{\mathbf{e}}^{(1)} = \hat{\boldsymbol{\imath}}$ and $\hat{\mathbf{e}}^{(2)} = \hat{\boldsymbol{\jmath}}$, the edges, $\mathbf{b}^{(1)}(t) = e^{2t}\hat{\mathbf{e}}^{(1)}$ and $\mathbf{b}^{(2)}(t) = e^{3t}\hat{\mathbf{e}}^{(2)}$, of the hunk of fluid never change direction. But their lengths change. The relative rate of change of length per unit time, $|\frac{d\,\mathbf{b}^{(k)}}{d\,t}(t)|/|\mathbf{b}^{(k)}(t)|$, is 2 for $\mathbf{b}^{(1)}$ and 3 for $\mathbf{b}^{(3)}$. In the figure on the right, the darker rectangle is the initial square. That is, the square with edges $\mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$. The lighter rectangle is that with edges $\mathbf{b}^{(k)}(t)$ for some $t$ a bit bigger than $t_0$.



**Example 1 - generalized.** The behaviour of Example 1 is typical of $\mathcal{V}$'s that are symmetric matrices, i.e. that obey $\mathcal{V}_{i,j} = \mathcal{V}_{j,i}$ for all $i, j$. Any $d \times d$ symmetric matrix (with real entries)

- has $d$ real eigenvalues
- has $d$ mutually orthogonal real unit eigenvectors.

Denote by $\lambda_k$, $1 \le k \le d$, the eigenvalues of $\mathcal{V}$ and choose $d$ mutually perpendicular real unit vectors, $\hat{\mathbf{e}}^{(k)}$, that obey $\mathcal{V}\hat{\mathbf{e}}^{(k)} = \lambda_k\hat{\mathbf{e}}^{(k)}$ for all $1 \le k \le d$. Then

$$\mathbf{b}^{(k)}(t) = e^{\lambda_k(t-t_0)}\ \hat{\mathbf{e}}^{(k)}$$

obeys

$$\tfrac{d\,\mathbf{b}^{(k)}}{d\,t}(t) = \lambda_k e^{\lambda_k(t-t_0)}\ \hat{\mathbf{e}}^{(k)} = e^{\lambda_k(t-t_0)}\ \mathcal{V}\hat{\mathbf{e}}^{(k)} = \mathcal{V}\mathbf{b}^{(k)}(t) \qquad \text{and} \qquad \mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$$

So $\mathbf{b}^{(k)}(t) = e^{\lambda_k(t-t_0)}\ \hat{\mathbf{e}}^{(k)}$ satisfies (IVP) for all $t$ and $1 \le k \le d$.

If we start, at time $t_0$, with a cube whose edges, $\hat{\mathbf{e}}^{(k)}$, are eigenvectors of $\mathcal{V}$, then as time progresses the edges, $\mathbf{b}^{(k)}(t)$, of the hunk of fluid never change direction. But their lengths change with the relative rate of change of length per unit time being $\lambda_k$ for edge number $k$. This rate of change may be positive (the edge grows with time) or negative (the edge shrinks in time) depending on the sign of $\lambda_k$. The volume of the hunk of fluid at time $t$ is $V(t) = e^{\lambda_1(t-t_0)} \cdots e^{\lambda_d(t-t_0)}$. The relative rate of change of volume per unit time is $V'(t)/V(t) = \lambda_1 \cdots + \lambda_d$, the sum of the $d$ eigenvalues. The sum of the eigenvalues of any $d \times d$ matrix $\mathcal{V}$ is given by its trace $\sum_{i=1}^d \mathcal{V}_{i,i}$. For the matrix (M)

$$\tfrac{V'(t_0)}{V(t_0)} = \sum_{i=1}^{d} \tfrac{\partial v_i}{\partial x_i}\big(\mathbf{x}_0, t_0\big) = \boldsymbol{\nabla} \cdot \mathbf{v}\big(\mathbf{x}_0, t_0\big)$$

So, at least when the matrix (M) is symmetric, the divergence $\boldsymbol{\nabla} \cdot \mathbf{v}\big(\mathbf{x}_0, t_0\big)$ gives the relative rate of change of volume per unit time for our tiny hunk of fluid at time $t_0$ and position $\mathbf{x}_0$.

**Example 1 - generalized yet again.**

For any $d \times d$ matrix $\mathcal{V}$, the solution of

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(t_0) = \mathbf{e}$$

is

$$\mathbf{b}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}$$

where the exponential of a $d \times d$ matrix $B$ is defined by the power series

$$e^B = \mathbb{1} + B + \tfrac{1}{2}B^2 + \tfrac{1}{3!}B^3 + \cdots = \sum_{n=0}^{\infty} \tfrac{1}{n!}B^n$$

with $\mathbb{1}$ denoting the $d \times d$ identity matrix. This sum converges for all $d \times d$ matrices $B$. Furthermore it easy to check, using the power series, that $e^{\mathcal{V}(t-t_0)}$ obeys $\frac{d}{dt}e^{\mathcal{V}(t-t_0)} = \mathcal{V}e^{\mathcal{V}(t-t_0)}$ and is the identity matrix when $t = t_0$. So $\mathbf{b}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}$ really does obey $\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t)$ and $\mathbf{b}(t_0) = \mathbf{e}$.

Pick any $d$ vectors $\mathbf{e}^{(k)}$, $1 \le k \le d$, and define $\mathbf{b}^{(k)}(t) = e^{\mathcal{V}(t-t_0)}\mathbf{e}^{(k)}$. Also let $E$ be the $d \times d$ matrix whose $k^{\text{th}}$ column is $\mathbf{e}^{(k)}$ and $E(t)$ be the $d \times d$ matrix whose $k^{\text{th}}$ column is

$\mathbf{b}^{(k)}(t)$. Then the volume of the parallelpiped with edges $\mathbf{e}^{(k)},\ 1 \le k \le d$, is $V(t_0) = \det E$ and the volume of the parallelpiped with edges $\mathbf{b}^{(k)}(t),\ 1 \le k \le d$, is

$$V(t) = \det E(t) = \det\left(e^{\mathcal{V}(t-t_0)}E\right) = \det\left(e^{\mathcal{V}(t-t_0)}\right)\det E = \det\left(e^{\mathcal{V}(t-t_0)}\right)V(t_0)$$

For any $d \times d$ matrix $B$, $\det e^B = e^{\operatorname{tr} B}$, so

$$V(t) = e^{(t-t_0)\operatorname{tr}\mathcal{V}}V(t_0) \qquad \Rightarrow \qquad \tfrac{V'(t_0)}{V(t_0)} = \operatorname{tr}\mathcal{V} = \sum_{i=1}^{d}\mathcal{V}_{i,i}$$

So, for any matrix (M) and any choice of $\hat{\mathbf{e}}^{(k)},\ 1 \le k \le d$, the divergence $\boldsymbol{\nabla}\cdot\mathbf{v}\big(\mathbf{x}_0, t_0\big)$ gives the relative rate of change of volume per unit time for our tiny hunk of fluid at time $t_0$ and position $\mathbf{x}_0$.

**Example 2:** $\mathbf{v}(x,y) = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$. In this example

$$\mathcal{V} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

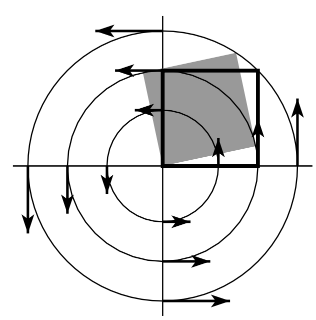
The solution to

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = (\beta_1, \beta_2) \quad \text{or equivalently} \quad \begin{array}{ll} b_1'(t) = -b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = b_1(t) & b_2(0) = \beta_2 \end{array}$$

is

$$\begin{array}{l} b_1(t) = \beta_1\cos t - \beta_2\sin t \\ b_2(t) = \beta_1\sin t + \beta_2\cos t \end{array} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\mathbf{b}(0)$$

The vector $\mathbf{b}(t)$ has the same length as $\mathbf{b}(0)$. The angle between $\mathbf{b}(t)$ and $\mathbf{b}(0)$ is $t$ radians. So, in this example, no matter what direction vectors $\hat{\mathbf{e}}^{(k)}$ we pick, the hunk of fluid just rotates at one radian per unit time. In the figure on the right, the outlined rectangle is the initial square. That is, the square with edges $\mathbf{b}^{(k)}(t_0) = \hat{\mathbf{e}}^{(k)}$. The shaded rectangle is that with edges $\mathbf{b}^{(k)}(t)$ for some $t$ a bit bigger than $t_0$.



**Example 2 - generalized.** The behaviour of example 2 is typical of $\mathcal{V}$'s that are antisymmetric matrices, i.e. that obey $\mathcal{V}_{i,j} = -\mathcal{V}_{j,i}$ for all $i, j$. As we have already observed, for any $d \times d$ matrix $\mathcal{V}$, the solution of $\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t)$, $\mathbf{b}(0) = \mathbf{e}$ is $\mathbf{b}(t) = e^{\mathcal{V}t}\mathbf{e}$. We now show that if $\mathcal{V}$ is a $3 \times 3$ antisymmetric matrix, then $e^{\mathcal{V}t}$ is a rotation. Assuming that $\mathcal{V}$ is not the zero matrix (in which case $e^{\mathcal{V}t}$ is the identity matrix for all $t$), we can find a number $\Omega > 0$ and a unit vector $\hat{\mathbf{k}} = (k_1, k_2, k_3)$ (not necessarily the standard unit vector parallel to the $z$–axis) such that

$$\mathcal{V} = \begin{bmatrix} 0 & -\Omega k_3 & \Omega k_2 \\ \Omega k_3 & 0 & -\Omega k_1 \\ -\Omega k_2 & \Omega k_1 & 0 \end{bmatrix} \tag{R}$$

This is easy. Because $\mathcal{V}$ is antisymmetric, all of the entries on its diagonal must be zero. Define $\Omega$ to be $\sqrt{\mathcal{V}_{1,2}^2 + \mathcal{V}_{1,3}^2 + \mathcal{V}_{2,3}^2}$ and $k_1 = -\mathcal{V}_{2,3}/\Omega$, $k_2 = \mathcal{V}_{1,3}/\Omega$, $k_3 = -\mathcal{V}_{1,2}/\Omega$. Also, let $\hat{\boldsymbol{\imath}}$ be any unit vector orthogonal to $\hat{\mathbf{k}}$ (again, not necessarily the standard one) and $\hat{\boldsymbol{\jmath}} = \hat{\mathbf{k}} \times \hat{\boldsymbol{\imath}}$. So $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$ is a right–handed system of three mutually perpendicular unit vectors.

Observe that, for any vector $\mathbf{e} = (e_1, e_2, e_3)$

$$\mathcal{V}\mathbf{e} = \begin{bmatrix} 0 & -\Omega k_3 & \Omega k_2 \\ \Omega k_3 & 0 & -\Omega k_1 \\ -\Omega k_2 & \Omega k_1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \Omega \begin{bmatrix} k_2 e_3 - k_3 e_2 \\ k_3 e_1 - k_1 e_3 \\ k_1 e_2 - k_2 e_1 \end{bmatrix} = \Omega \hat{\mathbf{k}} \times \mathbf{e}$$

In particular,

$$\begin{aligned} &\mathcal{V}\hat{\boldsymbol{\imath}} = \Omega\hat{\mathbf{k}} \times \hat{\boldsymbol{\imath}} = \Omega\hat{\boldsymbol{\jmath}} && \mathcal{V}\hat{\boldsymbol{\jmath}} = \Omega\hat{\mathbf{k}} \times \hat{\boldsymbol{\jmath}} = -\Omega\hat{\boldsymbol{\imath}} && \mathcal{V}\hat{\mathbf{k}} = \Omega\hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}} \\ &\mathcal{V}^2\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}\hat{\boldsymbol{\jmath}} = -\Omega^2\hat{\boldsymbol{\imath}} && \mathcal{V}^2\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}\hat{\boldsymbol{\imath}} = -\Omega^2\hat{\boldsymbol{\jmath}} && \mathcal{V}^2\hat{\mathbf{k}} = \mathcal{V}\vec{\mathbf{0}} = \vec{\mathbf{0}} \\ &\mathcal{V}^3\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}^2\hat{\boldsymbol{\jmath}} = -\Omega^3\hat{\boldsymbol{\jmath}} && \mathcal{V}^3\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}^2\hat{\boldsymbol{\imath}} = \Omega^3\hat{\boldsymbol{\imath}} && \mathcal{V}^3\hat{\mathbf{k}} = \mathcal{V}^2\vec{\mathbf{0}} = \vec{\mathbf{0}} \\ &\mathcal{V}^4\hat{\boldsymbol{\imath}} = \Omega\mathcal{V}^3\hat{\boldsymbol{\jmath}} = \Omega^4\hat{\boldsymbol{\imath}} && \mathcal{V}^4\hat{\boldsymbol{\jmath}} = -\Omega\mathcal{V}^3\hat{\boldsymbol{\imath}} = \Omega^4\hat{\boldsymbol{\jmath}} && \mathcal{V}^4\hat{\mathbf{k}} = \mathcal{V}^3\vec{\mathbf{0}} = \vec{\mathbf{0}} \end{aligned}$$

and so on. For all odd $n \ge 1$,

$$\mathcal{V}^n\hat{\boldsymbol{\imath}} = (-1)^{(n-1)/2}\Omega^n\hat{\boldsymbol{\jmath}} \qquad \mathcal{V}^n\hat{\boldsymbol{\jmath}} = -(-1)^{(n-1)/2}\Omega^n\hat{\boldsymbol{\imath}} \qquad \mathcal{V}^n\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

and all even $n \ge 2$,

$$\mathcal{V}^n\hat{\boldsymbol{\imath}} = (-1)^{n/2}\Omega^n\hat{\boldsymbol{\imath}} \qquad \mathcal{V}^n\hat{\boldsymbol{\jmath}} = (-1)^{n/2}\Omega^n\hat{\boldsymbol{\jmath}} \qquad \mathcal{V}^n\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Hence

$$\begin{aligned} e^{\mathcal{V}t}\hat{\boldsymbol{\imath}} &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathcal{V}t)^n\hat{\boldsymbol{\imath}} = \sum_{n \text{ even}} \tfrac{(-1)^{n/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\imath}} + \sum_{n \text{ odd}} \tfrac{(-1)^{(n-1)/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\jmath}} = \cos(\Omega t)\,\hat{\boldsymbol{\imath}} + \sin(\Omega t)\,\hat{\boldsymbol{\jmath}} \\ e^{\mathcal{V}t}\hat{\boldsymbol{\jmath}} &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathcal{V}t)^n\hat{\boldsymbol{\jmath}} = \sum_{n \text{ even}} \tfrac{(-1)^{n/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\jmath}} - \sum_{n \text{ odd}} \tfrac{(-1)^{(n-1)/2}}{n!}(\Omega t)^n\hat{\boldsymbol{\imath}} = -\sin(\Omega t)\,\hat{\boldsymbol{\imath}} + \cos(\Omega t)\,\hat{\boldsymbol{\jmath}} \\ e^{\mathcal{V}t}\hat{\mathbf{k}} &= \sum_{n=0}^{\infty} \tfrac{1}{n!}(\mathcal{V}t)^n\hat{\mathbf{k}} = \hat{\mathbf{k}} \end{aligned}$$

So $e^{\mathcal{V}t}$ is rotation by an angle $\Omega t$ about the axis $\hat{\mathbf{k}}$.

Whether or not the matrix (M) is antisymmetric, the matrix with entries

$$A_{i,j} = \tfrac{1}{2}\big(\mathcal{V}_{i,j} - \mathcal{V}_{j,i}\big)$$

is. When (M) is antisymmetric, $A$ and $\mathcal{V}$ coincide. The matrix $A$ is

$$A = \frac{1}{2}\begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2}(\mathbf{x}_0,t_0) - \frac{\partial v_2}{\partial x_1}(\mathbf{x}_0,t_0) & \frac{\partial v_1}{\partial x_3}(\mathbf{x}_0,t_0) - \frac{\partial v_3}{\partial x_1}(\mathbf{x}_0,t_0) \\ -\frac{\partial v_1}{\partial x_2}(\mathbf{x}_0,t_0) + \frac{\partial v_2}{\partial x_1}(\mathbf{x}_0,t_0) & 0 & \frac{\partial v_2}{\partial x_3}(\mathbf{x}_0,t_0) - \frac{\partial v_3}{\partial x_2}(\mathbf{x}_0,t_0) \\ -\frac{\partial v_1}{\partial x_3}(\mathbf{x}_0,t_0) + \frac{\partial v_3}{\partial x_1}(\mathbf{x}_0,t_0) & -\frac{\partial v_2}{\partial x_3}(\mathbf{x}_0,t_0) + \frac{\partial v_3}{\partial x_2}(\mathbf{x}_0,t_0) & 0 \end{bmatrix}$$

Comparing this with (R), we see that

$$\Omega\hat{\mathbf{k}} = \tfrac{1}{2}\nabla \times \mathbf{v}\big(\mathbf{x}_0,t_0\big)$$

So, at least when the matrix (M) is antisymmetric, our tiny cube rotates about the axis with $\nabla \times \mathbf{v}\big(\mathbf{x}_0,t_0\big)$ at rate $\frac{1}{2}\big|\nabla \times \mathbf{v}\big(\mathbf{x}_0,t_0\big)\big|$.

**Remark.** In the generalization of Example 2, we only considered dimension 3. It is a nice exercise in eigenvalues and eigenvectors to handle general dimension. Here are the main facts about antisymmetric matrices with real entries that are used.

- ○ All eigenvalues of antisymmetric matrices are either zero or pure imaginary.
- ○ For antisymmetric matrices with real entries, the nonzero eigenvalues come in complex conjugate pairs. The corresponding eigenvectors may also be chosen to be complex conjugates.

Choose as basis vectors (like $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$ above)

- ○ the eigenvectors of eigenvalue 0 (they act like $\hat{\mathbf{k}}$ above)
- ○ the real and imaginary parts of each complex conjugate pair of eigenvectors (they act like $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ above)

**Resumé so far:**

We have now seen that

- when the matrix $\mathcal{V}$ defined in (M) is symmetric and the direction vectors $\hat{\mathbf{e}}^{(k)}$ of the cube are eigenvectors of $\mathcal{V}$, then, at time $t_0$, the hunk of fluid is not changing orientation but is changing volume at instantaneous relative rate $\nabla \cdot \mathbf{v}\big(\mathbf{x}_0,t_0\big)$.
- when the matrix $\mathcal{V}$ defined in (M) is antisymmetric, then, at time $t_0$, the hunk of fluid is not changing shape or size but is rotating about the axis $\nabla \times \mathbf{v}\big(\mathbf{x}_0,t_0\big)$ at rate $\frac{1}{2}\big|\nabla \times \mathbf{v}\big(\mathbf{x}_0,t_0\big)\big|$. For this reason, $\nabla \times \mathbf{v}$ is often referred to as a "vorticity" meter.

**The general case:**

Now consider a general $\mathcal{V}$. It can always be written as the sum

$$\mathcal{V} = S + A$$

of a symmetric and an antisymmetric matrix. Just define

$$S_{i,j} = \tfrac{1}{2}\big(\mathcal{V}_{i,j} + \mathcal{V}_{j,i}\big) \qquad A_{i,j} = \tfrac{1}{2}\big(\mathcal{V}_{i,j} - \mathcal{V}_{j,i}\big)$$

As we have already observed, the solution of

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = \mathbf{e}$$

is

$$\mathbf{b}(t) = e^{\mathcal{V}t}\mathbf{e} = e^{(A+S)t}\mathbf{e}$$

If $S$ and $A$ were ordinary numbers, we would have $e^{(A+S)t} = e^{At}e^{St}$. But for matrices this need not be the case, unless $S$ and $A$ happen to commute. For arbitrary matrices, it is still true that

$$e^{(A+S)t} = \lim_{n\to\infty}\Big[e^{At/n}e^{St/n}\Big]^n$$

This is called the Lie product formula. It shows that our tiny hunk of fluid mixes together the behaviours of $A$ and $S$, scaling a bit, then rotating a bit, then scaling a bit and so on.

**Example 3:** $\mathbf{v}(x,y) = 2y\hat{\boldsymbol{\imath}}$.

In this example

$$\mathcal{V} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = S + A \qquad \text{with} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The solution to the full flow

$$\mathbf{b}'(t) = \mathcal{V}\mathbf{b}(t) \quad \mathbf{b}(0) = (\beta_1,\beta_2) \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = 2b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = 0 & b_2(0) = \beta_2 \end{matrix}$$

is

$$\begin{matrix} b_1(t) = \beta_1 + 2\beta_2 t \\ b_2(t) = \beta_2 \end{matrix} \quad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}\mathbf{b}(0)$$

The solution to the $S$ part of the flow

$$\mathbf{b}'(t) = S\mathbf{b}(t) \quad \mathbf{b}(0) = (\beta_1,\beta_2) \quad \text{or equivalently} \quad \begin{matrix} b_1'(t) = b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = b_1(t) & b_2(0) = \beta_2 \end{matrix}$$

is

$$b_1(t) = \beta_1 \cosh t + \beta_2 \sinh t \qquad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \mathbf{b}(0)$$
$$b_2(t) = \beta_1 \sinh t + \beta_2 \cosh t$$

The eigenvectors of $S$ are

$$\hat{\mathbf{e}}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \hat{\mathbf{e}}^{(2)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The corresponding eigenvalues are $+1$ and $-1$. The eigenvectors obey

$$e^{St}\hat{\mathbf{e}}^{(1)} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \hat{\mathbf{e}}^{(1)} = e^t \hat{\mathbf{e}}^{(1)} \qquad e^{St}\hat{\mathbf{e}}^{(2)} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \hat{\mathbf{e}}^{(2)} = e^{-t} \hat{\mathbf{e}}^{(2)}$$

Under the $S$ part of the flow $\hat{\mathbf{e}}^{(1)}$ scales by a factor of $e^t$, which is bigger than one for $t > 0$ and $\hat{\mathbf{e}}^{(2)}$ scales by a factor of $e^{-t}$, which is smaller than one for $t > 0$.

The solution to the $A$ part of the flow

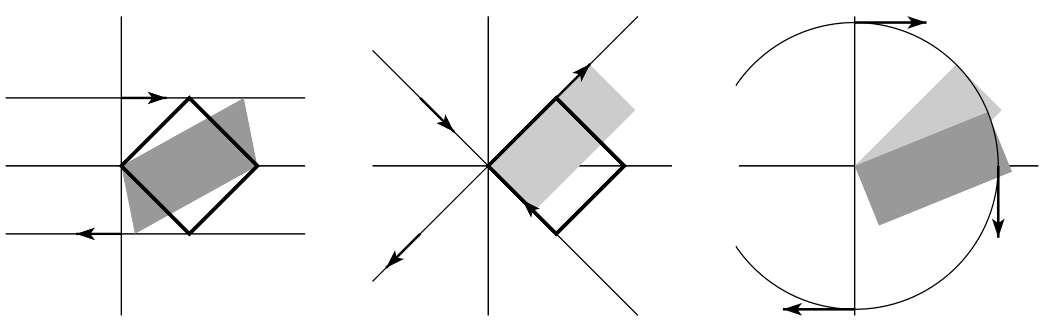
$$\mathbf{b}'(t) = A\mathbf{b}(t) \quad \mathbf{b}(0) = (\beta_1, \beta_2) \quad \text{or equivalently} \quad \begin{array}{ll} b_1'(t) = b_2(t) & b_1(0) = \beta_1 \\ b_2'(t) = -b_1(t) & b_2(0) = \beta_2 \end{array}$$

is

$$b_1(t) = \beta_1 \cos t + \beta_2 \sin t \qquad \text{or equivalently} \quad \mathbf{b}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{b}(0)$$
$$b_2(t) = -\beta_1 \sin t + \beta_2 \cos t$$

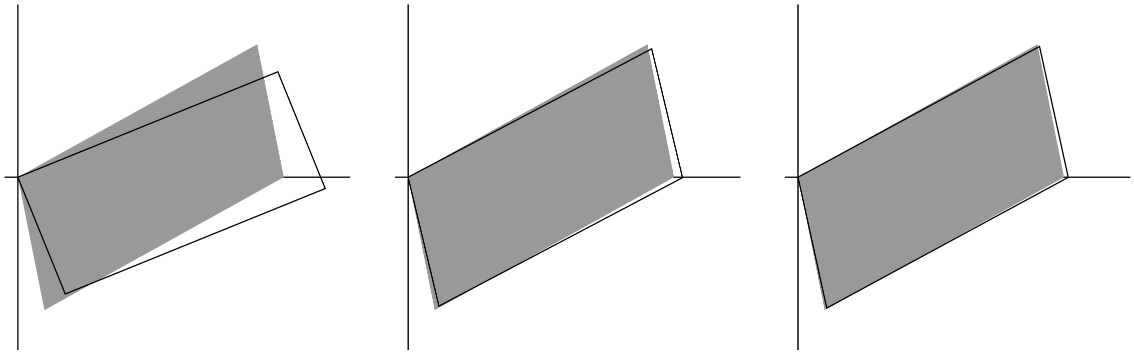
The $A$ part of the flow rotates clockwise about the origin at one radian per unit time.

Here are some figures. The first shows a square with edges $\hat{\mathbf{e}}^{(1)}$, $\hat{\mathbf{e}}^{(2)}$ and its image under the full flow $t = 0.4$ later. Under this full flow $\hat{\mathbf{e}}^{(k)} \to e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)}$. The second shows its image under 0.4 time units of the $S$–flow (that is, $\hat{\mathbf{e}}^{(k)} \to e^{0.4S}\hat{\mathbf{e}}^{(k)}$). The third applies 0.4 time units of the $A$–flow to the shaded rectangle of the middle figure. So the lightly shaded rectangle of the third figure has edges $e^{0.4S}\hat{\mathbf{e}}^{(k)}$ and the darkly shaded rectangle has edges $e^{0.4A}e^{0.4S}\hat{\mathbf{e}}^{(k)}$.



$\hat{\mathbf{e}}^{(k)}$ and $e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)}$ $\qquad$ $\hat{\mathbf{e}}^{(k)}$ and $e^{0.4S}\hat{\mathbf{e}}^{(k)}$ $\qquad$ $e^{0.4S}\hat{\mathbf{e}}^{(k)}$ and $e^{0.4A}e^{0.4S}\hat{\mathbf{e}}^{(k)}$

Of course $e^{0.4A}e^{0.4S}\hat{\mathbf{e}}^{(k)}$ is not a very good approximation for $e^{0.4(A+S)}\hat{\mathbf{e}}^{(k)}$. It is much better to take $\left[e^{0.4A/n}e^{0.4S/n}\right]^n\hat{\mathbf{e}}^{(k)}$ with $n$ large. Each of the following figures shows two parallelepipeds. In each, the shaded region has edges $e^{0.4\mathcal{V}}\hat{\mathbf{e}}^{(k)} = e^{0.4(A+S)}\hat{\mathbf{e}}^{(k)}$ and the outlined region has edges $\left[e^{0.4A/n}e^{0.4S/n}\right]^n\hat{\mathbf{e}}^{(k)}$.



$n = 1$ $\qquad$ $n = 5$ $\qquad$ $n = 10$