Faraday's Law

Faraday's Law is the following. Let S be an oriented surface with boundary C. Let $\mathbf{E}$ and $\mathbf{H}$ be the (time dependent) electric and magnetic fields and define

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = \text{voltage around } C \qquad \iint_S \mathbf{H}\cdot\hat{\mathbf{n}}\,dS = \text{magnetic flux through } S$$

Then the voltage around C is the negative of the rate of change of the magnetic flux through S. As an equation, Faraday's Law is

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\tfrac{\partial}{\partial t}\iint_S \mathbf{H}\cdot\hat{\mathbf{n}}\,dS$$

By Stokes' Theorem

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla}\times\mathbf{E})\cdot\hat{\mathbf{n}}\,dS$$

so

$$\iint_S \left(\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{\partial\mathbf{H}}{\partial t}\right)\cdot\hat{\mathbf{n}}\,dS = 0$$

This is true for all surfaces S. So the integrand, assuming that it is continuous, must be zero. To see this, let $\mathbf{G} = \left(\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{\partial\mathbf{H}}{\partial t}\right)$. Suppose that $\mathbf{G}(\mathbf{x}_0) \neq 0$. Pick a unit vector $\hat{\mathbf{n}}$ in the direction of $\mathbf{G}(\mathbf{x}_0)$. Let S be a very small flat disk centered on $\mathbf{x}_0$ with normal $\hat{\mathbf{n}}$ (the vector we picked). Then $\mathbf{G}(\mathbf{x}_0)\cdot\hat{\mathbf{n}} > 0$ and, by continuity, $\mathbf{G}(\mathbf{x})\cdot\hat{\mathbf{n}} > 0$ for all $\mathbf{x}$ on S, if we have picked S small enough. Then $\iint_S \left(\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{\partial\mathbf{H}}{\partial t}\right)\cdot\hat{\mathbf{n}}\,dS > 0$, which is a contradiction. So we conclude that

$$\boldsymbol{\nabla}\times\mathbf{E} + \tfrac{\partial\mathbf{H}}{\partial t} = 0$$

This is one of Maxwell's equations.