Surface and Flux Integrals

Parametrized Surfaces. If the surface is paramtrized by $\mathbf{r}(u,v)$, then

$$dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \, du\, dv \qquad \hat{\mathbf{n}} dS = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \, du\, dv$$

Graphs. If the surface is $z = f(x,y)$ or $x = g(y,z)$ or $y = h(x,z)$ then

$$\begin{aligned} dS &= \sqrt{1 + \left(\tfrac{\partial f}{\partial x}\right)^2 + \left(\tfrac{\partial f}{\partial y}\right)^2} \, dx\, dy & \hat{\mathbf{n}} dS &= \pm\Big[-\tfrac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \tfrac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big] dx\, dy \\ &= \sqrt{1 + \left(\tfrac{\partial g}{\partial y}\right)^2 + \left(\tfrac{\partial g}{\partial z}\right)^2} \, dy\, dz & &= \pm\Big[\hat{\boldsymbol{\imath}} - \tfrac{\partial g}{\partial y}\hat{\boldsymbol{\jmath}} - \tfrac{\partial g}{\partial z}\hat{\mathbf{k}}\Big] dy\, dz \\ &= \sqrt{1 + \left(\tfrac{\partial h}{\partial x}\right)^2 + \left(\tfrac{\partial h}{\partial z}\right)^2} \, dx\, dz & &= \pm\Big[-\tfrac{\partial h}{\partial x}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \tfrac{\partial h}{\partial z}\hat{\mathbf{k}}\Big] dx\, dz \end{aligned}$$

Level Surfaces. If the surface is $G(x,y,z) = 0$, then

$$\begin{aligned} dS &= \left|\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\mathbf{k}}}\right| dx\, dy = \left|\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\boldsymbol{\imath}}}\right| dy\, dz = \left|\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\boldsymbol{\jmath}}}\right| dx\, dz \\ \hat{\mathbf{n}} dS &= \pm\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\mathbf{k}}} dx\, dy = \pm\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\boldsymbol{\imath}}} dy\, dz = \pm\frac{\boldsymbol{\nabla} G}{\boldsymbol{\nabla} G \cdot \hat{\boldsymbol{\jmath}}} dx\, dz \end{aligned}$$