Flux Integral Example

Problem: Evaluate $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ where $\mathbf{F} = x^4\hat{\boldsymbol{\imath}} + 2y^2\hat{\boldsymbol{\jmath}} + z\hat{\mathbf{k}}$, S is the half of the surface $\frac{1}{4}x^2 + \frac{1}{9}y^2 + z^2 = 1$ with $z \ge 0$ and $\hat{\mathbf{n}}$ is the upward unit normal.

Solution 1. Parametrize the half-ellipsoid

$$x(\theta,\phi) = 2\cos\theta\sin\phi \qquad y(\theta,\phi) = 3\sin\theta\sin\phi \qquad z(\theta,\phi) = \cos\phi$$

with (θ,ϕ) running over $0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/2$. Then

$$\begin{aligned}
\left(\tfrac{\partial x}{\partial\theta}, \tfrac{\partial y}{\partial\theta}, \tfrac{\partial z}{\partial\theta}\right) &= (-2\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0)\\
\left(\tfrac{\partial x}{\partial\phi}, \tfrac{\partial y}{\partial\phi}, \tfrac{\partial z}{\partial\phi}\right) &= (2\cos\theta\cos\phi, 3\sin\theta\cos\phi, -\sin\phi)\\
\hat{\mathbf{n}}dS &= -\left(\tfrac{\partial x}{\partial\theta}, \tfrac{\partial y}{\partial\theta}, \tfrac{\partial z}{\partial\theta}\right) \times \left(\tfrac{\partial x}{\partial\phi}, \tfrac{\partial y}{\partial\phi}, \tfrac{\partial z}{\partial\phi}\right)d\theta d\phi\\
&= -(-3\cos\theta\sin^2\phi, -2\sin\theta\sin^2\phi, -6\sin\phi\cos\phi)d\theta d\phi\\
\mathbf{F} &= 2^4\cos^4\theta\sin^4\phi\,\hat{\boldsymbol{\imath}} + 2\times 3^2\sin^2\theta\sin^2\phi\,\hat{\boldsymbol{\jmath}} + \cos\phi\,\hat{\mathbf{k}}\\
\mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \Big[3\times 2^4\cos^5\theta\sin^6\phi + 2\times 3^2\times 2\sin^3\theta\sin^4\phi + 6\sin\phi\cos^2\phi\Big]\,d\theta\,d\phi
\end{aligned}$$

The extra minus sign in $\hat{\mathbf{n}}dS$ was put there to make the z component of $\hat{\mathbf{n}}$ positive. Since $\int_0^{2\pi}\cos^m\theta\,d\theta = \int_0^{2\pi}\sin^m\theta\,d\theta = 0$ for all odd integers m

$$\begin{aligned}
\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_0^{\pi/2}d\phi\int_0^{2\pi}d\theta\ 6\sin\phi\cos^2\phi = 12\pi\int_0^{\pi/2}d\phi\ \sin\phi\cos^2\phi = 12\pi\big[-\tfrac{1}{3}\cos^3\phi\big]_0^{\pi/2}\\
&= \boxed{4\pi}
\end{aligned}$$

Solution 2. Parametrize the half-ellipsoid

$$x(r,\theta) = 2r\cos\theta \qquad y(r,\theta) = 3r\sin\theta \qquad z(r,\theta) = \sqrt{1-r^2}$$

with (r,θ) running over $0 \le \theta \le 2\pi,\ 0 \le r \le 1$. Then

$$\begin{aligned}
\left(\tfrac{\partial x}{\partial\theta}, \tfrac{\partial y}{\partial\theta}, \tfrac{\partial z}{\partial\theta}\right) &= (-2r\sin\theta, 3r\cos\theta, 0)\\
\left(\tfrac{\partial x}{\partial r}, \tfrac{\partial y}{\partial r}, \tfrac{\partial z}{\partial r}\right) &= \left(2\cos\theta, 3\sin\theta, -\tfrac{r}{\sqrt{1-r^2}}\right)\\
\hat{\mathbf{n}}dS &= -\left(\tfrac{\partial x}{\partial\theta}, \tfrac{\partial y}{\partial\theta}, \tfrac{\partial z}{\partial\theta}\right) \times \left(\tfrac{\partial x}{\partial r}, \tfrac{\partial y}{\partial r}, \tfrac{\partial z}{\partial r}\right)dr\,d\theta\\
&= -\left(-\tfrac{3r^2\cos\theta}{\sqrt{1-r^2}}, -\tfrac{2r^2\sin\theta}{\sqrt{1-r^2}}, -6r\right)dr\,d\theta\\
\mathbf{F} &= 2^4r^4\cos^4\theta\,\hat{\boldsymbol{\imath}} + 2\times 3^2r^2\sin^2\theta\,\hat{\boldsymbol{\jmath}} + \sqrt{1-r^2}\,\hat{\mathbf{k}}\\
\mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \Big[3\times 2^4\tfrac{r^6}{\sqrt{1-r^2}}\cos^5\theta + 2^23^2\tfrac{r^4}{\sqrt{1-r^2}}\sin^3\theta + 6r\sqrt{1-r^2}\Big]\,dr\,d\theta
\end{aligned}$$

The extra minus sign in $\hat{\mathbf{n}}dS$ was put there to make the z component of $\hat{\mathbf{n}}$ positive. Since $\int_0^{2\pi} \cos^m\theta\, d\theta = \int_0^{2\pi} \sin^m\theta\, d\theta = 0$ for all odd integers m

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \int_0^1 dr \int_0^{2\pi} d\theta\ 6r\sqrt{1-r^2} = 12\pi \int_0^1 dr\ r\sqrt{1-r^2} = 12\pi\big[-\tfrac{1}{3}(1-r^2)^{3/2}\big]_0^1$$
$$= \boxed{4\pi}$$

Solution 3. The surface is of the form $G(x,y,z)=0$ with $G(x,y,z)=\frac{1}{4}x^2+\frac{1}{9}y^2+z^2-1$. Hence

$$\hat{\mathbf{n}}dS = \frac{\nabla G}{\nabla G\cdot\hat{\mathbf{k}}}dx\,dy = \frac{\frac{x}{2}\hat{\boldsymbol{\imath}}+\frac{2y}{9}\hat{\boldsymbol{\jmath}}+2z\hat{\mathbf{k}}}{2z}dx\,dy = \big(\tfrac{x}{z}\hat{\boldsymbol{\imath}}+\tfrac{y}{9z}\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}}\big)dx\,dy$$
$$\Rightarrow \qquad \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \big(\tfrac{x^5}{z}+\tfrac{2y^3}{9z}+z\big)dx\,dy$$

On S, $z=z(x,y)=\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$ and $\frac{x^2}{4}+\frac{y^2}{9}\le 1$, so

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} \big(\tfrac{x^5}{z(x,y)}+\tfrac{2y^3}{9z(x,y)}+z(x,y)\big)dx\,dy = \iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} z(x,y)\ dx\,dy$$

since both $\frac{x^5}{z(x,y)}$ and $\frac{2y^3}{9z(x,y)}$ are odd under $x\to -x,\ y\to -y$ and the domain of integration is even under $x\to -x,\ y\to -y$. So

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} \sqrt{1-\tfrac{x^2}{4}-\tfrac{y^2}{9}}\ dx\,dy$$

To evaluate this integral, make the change of variables

$$x = 2r\cos\theta \qquad y = 3r\sin\theta$$

Then the domain of integration, $\frac{x^2}{4}+\frac{y^2}{9}\le 1$ becomes $r^2\le 1$, the integrand $\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$ becomes $\sqrt{1-r^2}$ and $dx\,dy$ becomes

$$\left|\det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial r}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial r}\end{bmatrix}\right| dr\,d\theta = \left|\det\begin{bmatrix}2\cos\theta & -2r\sin\theta\\ 3\sin\theta & 3r\cos\theta\end{bmatrix}\right| dr\,d\theta = 6r\,dr\,d\theta$$

So

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \int_0^1 dr \int_0^{2\pi} d\theta\ 6r\sqrt{1-r^2} = 12\pi \int_0^1 dr\ r\sqrt{1-r^2} = 12\pi\big[-\tfrac{1}{3}(1-r^2)^{3/2}\big]_0^1$$
$$= \boxed{4\pi}$$

Solution 4. The surface is of the form $z = f(x,y)$ with $f(x,y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$. Hence

$$\hat{\mathbf{n}} dS = \Big[-\tfrac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \tfrac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big] dx\, dy = \left[\frac{\frac{x}{4}\hat{\boldsymbol{\imath}} + \frac{y}{9}\hat{\boldsymbol{\jmath}}}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + \hat{\mathbf{k}} \right] dx\, dy$$

$$\Rightarrow \qquad \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \left[\frac{\frac{x^5}{4} + \frac{2y^3}{9}}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \right] dx\, dy$$

As in Solution 3, by oddness,

$$\begin{aligned} \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS &= \iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} \left[\frac{\frac{x^5}{4}+\frac{2y^3}{9}}{\sqrt{\cdots}} + \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \right] dx\, dy \\ &= \iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}\, dx\, dy \end{aligned}$$

Now continue as in Solution 3.