Interpretation of Gradient, Divergence and Curl

Gradient

The rate of change of a function f per unit distance as you leave the point (x_0, y_0, z_0) moving in the direction of the unit vector $\hat{\mathbf{n}}$ is given by the directional derivative

$$D_{\hat{\mathbf{n}}} f(x_0, y_0, z_0) = \boldsymbol{\nabla} f(x_0, y_0, z_0) \cdot \hat{\mathbf{n}} = |\boldsymbol{\nabla} f(x_0, y_0, z_0)| \cos\theta$$

where θ is the angle between $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ and $\hat{\mathbf{n}}$. The angle θ which maximizes this rate of change is 0 since $\cos 0 = 1 \geq \cos\theta$ for all θ. The angle θ is zero when $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ and $\hat{\mathbf{n}}$ point in the same direction. This tells us that

- The direction of the vector $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ is the direction that gives the maximum rate of change of f per unit distance as you leave the (x_0, y_0, z_0).
- The length of the vector $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ is magnitude of the maximum rate of change of f per unit distance as you leave the (x_0, y_0, z_0).

Divergence

Let $B_\varepsilon(x_0, y_0, z_0)$ be a tiny ball centred on the point (x_0, y_0, z_0) and denote by $S_\varepsilon(x_0, y_0, z_0)$ its surface. Because $B_\varepsilon(x_0, y_0, z_0)$ is really small, $\boldsymbol{\nabla} \cdot \mathbf{v}$ is essentially constant in $B_\varepsilon(x_0, y_0, z_0)$ and we essentially have

$$\iiint_{B_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \cdot \mathbf{v}\ dV = \boldsymbol{\nabla} \cdot \mathbf{v}(x_0, y_0, z_0)\ \mathrm{Vol}\big(B_\varepsilon(x_0, y_0, z_0)\big)$$

By the divergence theorem, we also have

$$\iiint_{B_\varepsilon(x_0,y_0,z_0)} \boldsymbol{\nabla} \cdot \mathbf{v}\ dV = \iint_{S_\varepsilon(x_0,y_0,z_0)} \mathbf{v} \cdot \hat{\mathbf{n}}\ dS$$

Think of the vector field $\mathbf{v}$ as the velocity of a moving fluid of density one. We have already seen that the flux integral for a velocity field has the interpretation

$$\begin{aligned} \iint_{S_\varepsilon(x_0,y_0,z_0)} \mathbf{v} \cdot \hat{\mathbf{n}}\ dS &= \text{the rate at which fluid is leaving } B_\varepsilon(x_0, y_0, z_0) \text{ through } S_\varepsilon(x_0, y_0, z_0) \\ &= \text{the rate at which fluid is being created in } B_\varepsilon(x_0, y_0, z_0) \end{aligned}$$

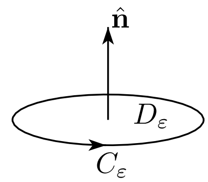
We conclude that

$$\boldsymbol{\nabla} \cdot \mathbf{v}(x_0, y_0, z_0) = \lim_{\varepsilon \to 0} \frac{\text{the rate at which fluid is being created in } B_\varepsilon(x_0, y_0, z_0)}{\mathrm{Vol}\big(B_\varepsilon(x_0, y_0, z_0)\big)}$$

This is called "the strength of the source at (x_0, y_0, z_0)".

Curl

Let $D_\varepsilon(x_0,y_0,z_0)$ be a tiny flat circular disk of radius ε centred on the point (x_0,y_0,z_0) and denote by $C_\varepsilon(x_0,y_0,z_0)$ its boundary circle. Let $\hat{\mathbf{n}}$ be a unit normal vector to D_ε. It tells us the orientation of D_ε. Give the circle C_ε the corresponding orientation using the right hand rule. That is, if the fingers of your right hand are pointing in the corresponding direction of motion along C_ε and your palm is facing D_ε, then the thumb is pointing in the direction $\hat{\mathbf{n}}$. Because $D_\varepsilon(x_0,y_0,z_0)$



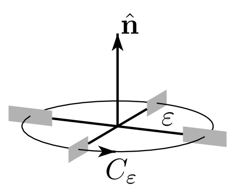
is really small, $\nabla\times\mathbf{v}$ is essentially constant on $D_\varepsilon(x_0,y_0,z_0)$ and we essentially have

$$\iint_{D_\varepsilon(x_0,y_0,z_0)} \nabla\times\mathbf{v}\cdot\hat{\mathbf{n}}\,dS = \nabla\times\mathbf{v}(x_0,y_0,z_0)\cdot\hat{\mathbf{n}}\ \mathrm{Area}\big(D_\varepsilon(x_0,y_0,z_0)\big) = \pi\varepsilon^2\ \nabla\times\mathbf{v}(x_0,y_0,z_0)\cdot\hat{\mathbf{n}}$$

By Stokes' theorem, we also have

$$\iint_{D_\varepsilon(x_0,y_0,z_0)} \nabla\times\mathbf{v}\cdot\hat{\mathbf{n}}\,dS = \oint_{C_\varepsilon(x_0,y_0,z_0)} \mathbf{v}\cdot d\mathbf{r}$$

Again, think of the vector field $\mathbf{v}$ as the velocity of a moving fluid. Then $\oint_{C_\varepsilon}\mathbf{v}\cdot d\mathbf{r}$ is called the circulation of $\mathbf{v}$ around C_ε. To measure the circulation experimentally, place a small paddle wheel in the fluid, with the axle of the paddle wheel pointing along $\hat{\mathbf{n}}$ and each of the paddles perpendicular to C_ε and centred on C_ε. Each paddle moves tangentially to C_ε. It would like to



move with the same speed as the tangential speed $\mathbf{v}\cdot\hat{\mathbf{t}}$ (where $\hat{\mathbf{t}}$ is the forward pointing unit tangent vector to C_ε at the location of the paddle) of the fluid at its location. But all paddles are rigidly fixed to the axle of the paddle wheel and so must all move with the same speed. That common speed will be the average value of $\mathbf{v}\cdot\hat{\mathbf{t}}$ around C_ε. If ds represents an element of arc length of C_ε, the average value of $\mathbf{v}\cdot\hat{\mathbf{t}}$ around C_ε is

$$\overline{v_T} = \tfrac{1}{2\pi\varepsilon}\oint_{C_\varepsilon}\mathbf{v}\cdot\hat{\mathbf{t}}\,ds = \tfrac{1}{2\pi\varepsilon}\oint_{C_\varepsilon}\mathbf{v}\cdot d\mathbf{r}$$

since $d\mathbf{r}$ has direction $\hat{\mathbf{t}}$ and length ds so that $d\mathbf{r} = \hat{\mathbf{t}}ds$. If the paddle wheel rotates at Ω radians per unit time, each paddle travels a distance $\Omega\varepsilon$ (remember that ε is the radius of C_ε) per unit time. That is $\overline{v_T} = \Omega\varepsilon$. Combining all this info,

$$\Omega\varepsilon = \overline{v_T} = \tfrac{1}{2\pi\varepsilon}\oint_{C_\varepsilon}\mathbf{v}\cdot d\mathbf{r} = \tfrac{1}{2\pi\varepsilon}\iint_{D_\varepsilon}\nabla\times\mathbf{v}\cdot\hat{\mathbf{n}}\,dS = \tfrac{1}{2\pi\varepsilon}\pi\varepsilon^2\ \nabla\times\mathbf{v}(x_0,y_0,z_0)\cdot\hat{\mathbf{n}}$$

so that

$$\Omega = \tfrac{1}{2}\nabla \times \mathbf{v}(x_0, y_0, z_0) \cdot \hat{\mathbf{n}}$$

The component of $\nabla \times \mathbf{v}(x_0, y_0, z_0)$ in any direction $\hat{\mathbf{n}}$ is twice the rate at which the paddle wheel turns when it is put into the fluid at (x_0, y_0, z_0) with its axle pointing in the direction $\hat{\mathbf{n}}$. The direction of $\nabla \times \mathbf{v}(x_0, y_0, z_0)$ is the axle direction which gives maximum rate of rotation and the magnitude of $\nabla \times \mathbf{v}(x_0, y_0, z_0)$ is twice that maximum rate of rotation. For this reason, $\nabla \times \mathbf{v}$ is often referred to as a "vorticity" meter.