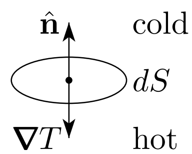
The Heat Equation

Let $T(x, y, z, t)$ be the temperature at time t at the point (x, y, z) in some body. The heat equation is the partial differential equation that describes the flow of heat energy and consequently the behaviour of T. We start by deriving the heat equation from two physical "laws", that we assume are valid:

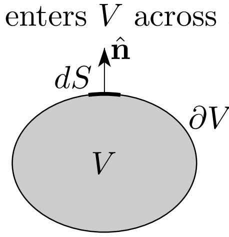
- The amount of heat energy required to raise the temperature of a body by ΔT degrees is $CM\Delta T$ where, M is the mass of the body and C is a positive physical constant determined by the material contained in the body. It is called the specific heat of the body.
- Think of heat energy as a fluid. It has velocity field $-k\boldsymbol{\nabla}T(x, y, z, t)$, where k is another positive physical constant called the thermal conductivity of the body. That is, the rate at which heat is conducted across an element of surface area dS at (x, y, z) in the direction of its unit normal $\hat{\mathbf{n}}$ is given by $-k\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}T(x, y, z, t)\, dS$ at time t. So heat flows in the direction opposite to the temperature gradient. For example, in the figure



  the temperature gradient, which points in the direction of increasing temperature, is opposite $\hat{\mathbf{n}}$. Consequently the flow rate $-k\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}T(x, y, z, t)\, dS$ is positive, indicating flow in the direction of $\hat{\mathbf{n}}$. This is just what you would expect – heat flows from hot regions to cold regions. Also the rate of flow increases as the magnitude of the temperature gradient increases. This also makes sense.

Let V be any three dimensional region in the body and denote by ∂V the surface of V and by $\hat{\mathbf{n}}$ the outward normal to ∂V. The amount of heat that **enters** V across an infinitesmal piece dS of ∂V in an infinitesmal time interval dt is $-\big(-k\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}T(x, y, z, t)\, dS\big)\, dt$, where $\hat{\mathbf{n}}$ is the outward normal to ∂V. The amount of heat that enters V across all of ∂V in the time interval dt is

$$\iint_{\partial V} k\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}T(x, y, z, t)\, dS\, dt$$



In this same time interval the temperature at a point (x, y, z) in V changes by $\frac{\partial T}{\partial t}(x, y, z, t)\, dt$. If the density of the body at (x, y, z) is $\rho(x, y, z)$, the amount of heat energy required to increase the temperature of an infinitesmal volume dV of the body centred at (x, y, z) by $\frac{\partial T}{\partial t}(x, y, z, t)\, dt$ is $C\rho\, dV\, \frac{\partial T}{\partial t}(x, y, z, t)\, dt$. The amount of heat energy required to increase the temperature by $\frac{\partial T}{\partial t}(x, y, z, t)\, dt$ at all points (x, y, z) in V is

$$\iiint_V C\rho \tfrac{\partial T}{\partial t}(x, y, z, t)\, dV\, dt$$

Assuming that the body is not generating or destroying heat itself, this must be same as the amount of heat that entered V in the time interval dt. That is

$$\iint_{\partial V} k\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} T\, dS\, dt = \iiint_V C\rho \tfrac{\partial T}{\partial t}\, dV\, dt$$

Cancelling the common factors of dt and applying the divergence theorem to the left hand side gives

$$\iiint_V k\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T\, dV = \iiint_V C\rho \tfrac{\partial T}{\partial t}\, dV \Rightarrow \iiint_V \big[k\boldsymbol{\nabla}^2 T - C\rho \tfrac{\partial T}{\partial t}\big]\, dV = 0 \tag{1}$$

where $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian. This must be true for all volumes V in the body and for all times t. I claim that this forces

$$k\boldsymbol{\nabla}^2 T(x, y, z, t) - C\rho \tfrac{\partial T}{\partial t}(x, y, z, t) = 0$$

for all (x, y, z) in the body and all t. Suppose that to the contrary we had, for example, $k\boldsymbol{\nabla}^2 T(x_0, y_0, z_0, t_0) - C\rho \frac{\partial T}{\partial t}(x_0, y_0, z_0, t_0) > 0$ for some (x_0, y_0, z_0) in the body. Then, by continuity, we would have $k\boldsymbol{\nabla}^2 T(x, y, z, t_0) - C\rho \frac{\partial T}{\partial t}(x, y, z, t_0) > 0$ for all (x, y, z) in some small ball B centered on (x_0, y_0, z_0). Then, necessarily,

$$\iiint_B \big[k\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} T(x, y, z, t_0) - C\rho \tfrac{\partial T}{\partial t}(x, y, z, t_0)\big]\, dV > 0$$

which violates (1) for $V = B$. This completes our derivation of the heat equation, which is

$$\tfrac{\partial T}{\partial t}(x, y, z, t) = \kappa \boldsymbol{\nabla}^2 T(x, y, z, t)$$

where $\kappa = \frac{k}{C\rho}$ is called the thermal diffusivity.

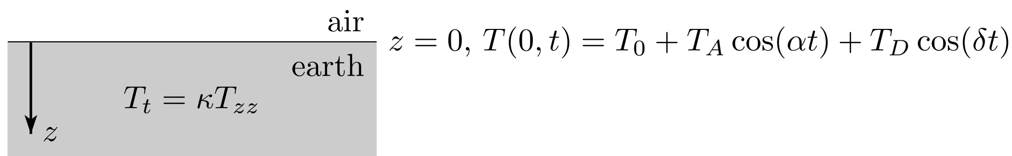
As an application, we look at the temperature a bit below the surface of the Earth. For simplicity, we make the Earth flat and we assume that the temperature, T, depends only on time, t, and the vertical coordinate, z. Then the heat equation simplifies to

$$\tfrac{\partial T}{\partial t}(z, t) = \kappa \tfrac{\partial^2 T}{\partial z^2}(z, t) \tag{2}$$

We choose a coordinate system having the surface of the Earth at $z = 0$ and having z increase downward. We also assume that the temperature $T(0, t)$ at the surface of the Earth is primarily determined by solar heating and is given by

$$T(0, t) = T_0 + T_A \cos(\alpha t) + T_D \cos(\delta t) \tag{3}$$

Here T_0 is the long term average of the temperature at the surface of the Earth, $T_A \cos(\alpha t)$ gives seasonal temperature variations and $T_D \cos(\delta t)$ gives daily temperature variations.



We measure time in seconds so that $\delta = \frac{2\pi}{1\ \text{day}} = \frac{2\pi}{86{,}400\text{sec}}$ and $\alpha = \frac{2\pi}{1\ \text{year}} = \frac{2\pi}{3.15\times 10^7\text{sec}}$. Then $T_A \cos(\alpha t)$ has period one year and $T_D \cos(\delta t)$ has period one day. I will just quote the solution to (2) and (3). It is

$$T(z, t) = T_0 + T_A e^{-\sqrt{\frac{\alpha}{2\kappa}}\, z} \cos\left(\alpha t - \sqrt{\tfrac{\alpha}{2\kappa}}\ z\right) + T_D e^{-\sqrt{\frac{\delta}{2\kappa}}\, z} \cos\left(\delta t - \sqrt{\tfrac{\delta}{2\kappa}}\ z\right) \tag{4}$$

If you have taken a course in partial differential equations, you can find this solution by separation of variables. In any event, you can check that (4) satisfies both (2) and (3). For any z, the time average of $T(z, t)$ is T_0, the same as the average temperature at the surface $z = 0$. The term $T_A e^{-\sqrt{\frac{\alpha}{2\kappa}}\, z} \cos\left(\alpha t - \sqrt{\frac{\alpha}{2\kappa}}\ z\right)$

- oscillates in time with a period of one year, just like $T_A \cos(\alpha t)$
- has an amplitude $T_A e^{-\sqrt{\frac{\alpha}{2\kappa}}\, z}$ which is T_A at the surface and decreases exponentially as z increases. Increasing the depth z by a distance $\sqrt{\frac{2\kappa}{\alpha}}$ causes the amplitude to decrease by a factor of $\frac{1}{e}$.
- has a time lag of $\frac{z}{\sqrt{2\kappa\alpha}}$ with respect to $T_A \cos(\alpha t)$. The surface term $T_A \cos(\alpha t)$ takes its maximum value when $t = 0,\ \frac{2\pi}{\alpha},\ \frac{4\pi}{\alpha},\ \cdots$. At depth z, the corresponding term $T_A e^{-\sqrt{\frac{\alpha}{2\kappa}}\, z} \cos\left(\alpha t - \sqrt{\frac{\alpha}{2\kappa}}\ z\right)$ takes its maximum value when $\alpha t - \sqrt{\frac{\alpha}{2\kappa}}\ z = 0,\ 2\pi, 4\pi,\ \cdots$ so that $t = \frac{z}{\sqrt{2\kappa\alpha}},\ \frac{2\pi}{\alpha} + \frac{z}{\sqrt{2\kappa\alpha}},\ \frac{4\pi}{\alpha} + \frac{z}{\sqrt{2\kappa\alpha}},\ \cdots$.

Similarly, the term $T_D e^{-\sqrt{\frac{\delta}{2\kappa}}\, z} \cos\left(\delta t - \sqrt{\frac{\delta}{2\kappa}}\ z\right)$

- oscillates in time with a period of one day, just like $T_D \cos(\delta t)$
- has an amplitude which is T_D at the surface and decreases by a factor of $\frac{1}{e}$ for each increase of $\sqrt{\frac{2\kappa}{\delta}}$ in depth.
- has a time lag of $\frac{z}{\sqrt{2\kappa\delta}}$ with respect to $T_D \cos(\delta t)$.

For soil $\kappa \approx 0.005\ \text{cm}^2/\text{sec}$. This κ gives

$$\sqrt{\tfrac{2\kappa}{\alpha}} \approx 2\text{m} \qquad \sqrt{\tfrac{2\kappa}{\delta}} \approx 10\text{cm} \qquad \tfrac{z}{\sqrt{2\kappa\alpha}} \approx 0.3z\ \text{days} \qquad \tfrac{z}{\sqrt{2\kappa\delta}} \approx 0.3z\ \text{hours}$$

for z measured in centimeters.