Hyperbolic Trig Functions

The hyperbolic trig functions are defined by

$$\sinh x = \tfrac{e^x - e^{-x}}{2} \quad \cosh x = \tfrac{e^x + e^{-x}}{2} \quad \tanh x = \tfrac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\operatorname{csch} x = \tfrac{2}{e^x - e^{-x}} \quad \operatorname{sech} x = \tfrac{2}{e^x + e^{-x}} \quad \coth x = \tfrac{e^x + e^{-x}}{e^x - e^{-x}}$$

As their names suggest, these functions are very closely related to the trig functions. This relationship may be seen from the formulae

$$\sin x = \tfrac{e^{ix} - e^{-ix}}{2i} \quad \cos x = \tfrac{e^{ix} + e^{-ix}}{2} \quad \tan x = -i\tfrac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}$$
$$\csc x = \tfrac{2i}{e^{ix} - e^{-ix}} \quad \sec x = \tfrac{2}{e^{ix} + e^{-ix}} \quad \cot x = i\tfrac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}$$

(If you are not familiar with these formulae, see the handout entitled "Complex Numbers and Exponentials".) In particular

$$\sinh x = i\sin(-ix) \qquad \cosh x = \cos(-ix)$$

Consequently, the differentiation formulae for hyperbolic trig functions are almost identical to those for trig functions:

$$\tfrac{d}{dx}\sinh x = \cosh x \qquad \tfrac{d}{dx}\cosh x = \sinh x$$

They differ only by some sign changes. Similarly, for each trig identity there is a corresponding hyperbolic trig identity, which is also identical up to sign changes:

$$\cosh^2 x - \sinh^2 x = \tfrac{e^{2x}+2+e^{-2x}}{4} - \tfrac{e^{2x}-2+e^{-2x}}{4} = 1$$
$$\sinh 2x = \tfrac{e^{2x}-e^{-2x}}{2} = 2\sinh x\cosh x$$
$$\cosh 2x = \tfrac{e^{2x}+e^{-2x}}{2} = 2\cosh^2 x - 1$$

Example. Find $\int \sqrt{a^2+x^2}\,dx$.

Solution. The standard technique for integrating $\int \sqrt{a^2-x^2}\,dx$ is to substitute $x = a\sin t$ and exploit the trig identity $1-\sin^2 t = \cos^2 t$ to eliminate the square root. The analog here is to substitute $x = a\sinh t$ ($\sinh t$ is a strictly increasing function, so the change of variables is legitimate) and exploit $1+\sinh^2 t = \cosh^2 t$, which we do. Since $dx = a\cosh t\,dt$,

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\sinh^2 t}\;a\cosh t\,dt = \int \sqrt{a^2\cosh^2 t}\;a\cosh t\,dt$$
$$= \int a^2\cosh^2 t\,dt$$

Note that $\cosh t > 0$ for all t, so we have correctly taken the positive square root. The standard technique for integrating $\int \cos^2 t\,dt$ exploits the trig identity $\cos^2 t = \frac{1+\cos 2t}{2}$. To integrate $\int a^2\cosh^2 t\,dt$ we use $\cosh^2 t = \frac{1+\cosh 2t}{2}$.

$$\int a^2\cosh^2 t\,dt = a^2\int \tfrac{1+\cosh 2t}{2}\,dt = \tfrac{a^2}{2}\big[t + \tfrac{1}{2}\sinh 2t\big] + C$$

To get back to the original variable x, sub in $t = \sinh^{-1}\frac{x}{a}$ and use the identities $\sinh 2t = 2\sinh t\cosh t$ and $\cosh t = \sqrt{\sinh^2 t + 1}$.

$$\begin{aligned}\tfrac{a^2}{2}\big[t + \tfrac{1}{2}\sinh 2t\big] &= \tfrac{a^2}{2}\big[t + \sinh t\cosh t\big] = \tfrac{a^2}{2}t + \tfrac{a}{2}\sinh t\sqrt{a^2\sinh^2 t + a^2}\\ &= \tfrac{a^2}{2}\sinh^{-1}\tfrac{x}{a} + \tfrac{1}{2}x\sqrt{x^2+a^2}\end{aligned}$$

All together

$$\boxed{\int\sqrt{a^2+x^2}\,dx = \tfrac{a^2}{2}\sinh^{-1}\tfrac{x}{a} + \tfrac{1}{2}x\sqrt{x^2+a^2} + C}$$

In addition, the inverse hyperbolic trig function $\sinh^{-1}x$ can be explicitly expressed in terms of ln's. By definition, $y = \sinh^{-1}x$ is the unique solution of $\sinh y = x$, or

$$\tfrac{e^y - e^{-y}}{2} = x \quad\Rightarrow\quad e^y - e^{-y} = 2x \quad\Rightarrow\quad e^{2y} - 1 = 2xe^y \quad\Rightarrow\quad e^{2y} - 2xe^y - 1 = 0$$

Think of this as a quadratic equation in e^y, with x just being a given constant. The solution, by the high school formula for the solution of a quadratic equation is

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

As y is a real number, e^y must be a positive number and we must reject the negative sign. Thus

$$y = \sinh^{-1}x = \ln\big(x + \sqrt{x^2+1}\big)$$

and we may rewrite the above integral as

$$\boxed{\int\sqrt{a^2+x^2}\,dx = \tfrac{a^2}{2}\ln\big(x+\sqrt{x^2+a^2}\big) + \tfrac{1}{2}x\sqrt{x^2+a^2} + C'}$$

(with the new constant $C' = C - \frac{a^2}{2}\ln a$).