Review of Ordinary Differential Equations

Definition 1

(a) A **differential equation** is an equation for an unknown function that contains the derivatives of that unknown function. For example $y''(t)+y(t)=0$ is a differential equation for the unknown function $y(t)$.

(b) A differential equation is called an **ordinary differential equation** (often shortened to "ODE") if only ordinary derivatives appear. That is, if the unknown function has only a single independent variable. A differential equation is called a **partial differential equation** (often shortened to "PDE") if partial derivatives appear. That is, if the unknown function has more than one independent variable. For example $y''(t)+y(t)=0$ is an ODE while $\frac{\partial^2 u}{\partial t^2}(x,t)=c^2\frac{\partial^2 u}{\partial x^2}(x,t)$ is a PDE.

(c) The **order** of a differential equation is the order of the highest derivative that appears. For example $y''(t)+y(t)=0$ is a second order ODE.

(d) An ordinary differential equation that is of the form

$$a_0(t)y^{(n)}(t)+a_1(t)y^{(n-1)}(t)+\cdots+a_{n-1}(t)y'(t)+a_n(t)y(t)=F(t) \tag{1}$$

with given coefficient functions $a_0(t)$, $\cdots$, $a_n(t)$ and $F(t)$ is said to be **linear**. Otherwise, the ODE is said to be **nonlinear**. For example, $y'(t)^2+y(t)=0$, $y'(t)y''(t)+y(t)=0$ and $y'(t)=e^{y(t)}$ are all nonlinear.

(e) The ODE (1) is said to have **constant coefficients** if the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_n(t)$ are all contants. Otherwise, it is said to have **variable coefficients**. For example, the ODE $y''(t)+7y(t)=\sin t$ is constant coefficient, while $y''(t)+ty(t)=\sin t$ is variable coefficient.

(f) The ODE (1) is said to be **homogeneous** if $F(t)$ is identically zero. Otherwise, it is said to be **inhomogeneous** or **nonhomogeneous**. For example, the ODE $y''(t)+7y(t)=0$ is homogeneous, while $y''(t)+7y(t)=\sin t$ is inhomogeneous. A homogeneous ODE always has the trivial solution $y(t)=0$.

(g) An **initial value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given initial conditions, like $y(0)=1$, $y'(0)=0$.

(h) A **boundary value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given boundary conditions, like $y(0)=0$, $y(1)=0$.

The following theorem gives the form of solutions to the ODE (1).

Theorem 2 *Assume that the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_{n-1}(t)$, $a_n(t)$ and $F(t)$ are reasonably smooth, bounded functions and that $a_0(t)$ is not zero.*

(a) The general solution to the ODE (1) is of the form

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t) \tag{2}$$

where

- *n is the order of the ODE (1)*
- *$y_p(t)$ is any solution to (1)*
- *C_1, C_2, $\cdots$, C_n are arbitrary constants*
- *y_1, y_2, $\cdots$, y_n are n independent solutions to the homogenous equation*

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = 0$$

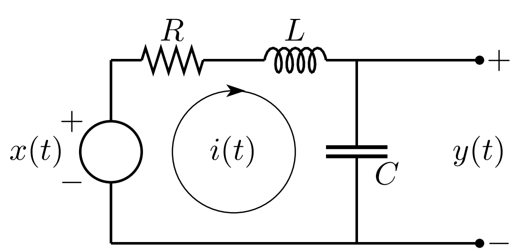
  associated to (1). "Independent" just means that no y_i can be written as a linear combination of the other y_j's. For example, $y_1(t)$ cannot be expressed in the form $d_2y_2(t) + \cdots + d_ny_n(t)$.

In (2), y_p is called the "particular solution" and $C_1y_1(t)+C_2y_2(t)+\cdots+C_ny_n(t)$ is called the "complementary solution".

(b) Given any constants b_0, $\cdots$, b_{n-1} there is exactly one function $y(t)$ that obeys the ODE (1) and the initial conditions

$$y(0) = b_0 \qquad y'(0) = b_1 \qquad \cdots \qquad y^{(n-1)}(0) = b_{n-1}$$

Example 3 (The RLC circuit) As an example of the most commonly used techniques for solving linear, constant coefficient ODE's, we consider the RLC circuit



We're going to think of the voltage $x(t)$ as an input signal, and the voltage $y(t)$ as an output signal. The goal is to determine the output signal produced by a given input signal. In the notes "The RLC Circuit", the ODE

$$LCy''(t) + RCy'(t) + y(t) = x(t) \tag{3}$$

is derived. As a concrete example, we'll take an ac voltage source and choose the origin of time so that $x(0) = 0$, $x(t) = E_0 \sin(\omega t)$. Then the differential equation becomes

$$LCy''(t) + RCy'(t) + y(t) = E_0 \sin(\omega t) \tag{4}$$

This is a second order, linear, constant coefficient ODE. So we know, from Theorem 2, that the general solution is of the form $y_p(t) + C_1 y_1(t) + C_2 y_2(t)$, where

- $y_p(t)$, the particular solution, is any one solution to (4),
- C_1, C_2 are arbitrary constants and
- $y_1(t)$, $y_2(t)$ are any two independent solutions of the corresponding homogeneous equation

$$LCy''(t) + RCy'(t) + y(t) = 0 \tag{4$_\mathrm{h}$}$$

So to find the general solution to (4), we need to find three functions: $y_1(t)$, $y_2(t)$ and $y_p(t)$.

Finding $y_1(t)$ and $y_2(t)$: The best way to find y_1 and y_2 is to guess them. Any solution, $y_h(t)$, of (4_h) has to have the property that $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ cancel each other out for all t. We choose our guess so that $y_h(t)$, $y_h'(t)$ and $y_h''(t)$ are all proportional to a single function of t. Then it will be easy to see if $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ all cancel. Hence we try $y_h(t) = e^{rt}$, with the constant r to the determined. This guess is a solution of (4_h) if and only if

$$LCr^2e^{rt} + RCre^{rt} + e^{rt} = 0 \iff LCr^2 + RCr + 1 = 0 \iff r = \tfrac{-RC\pm\sqrt{R^2C^2-4LC}}{2LC} \equiv r_{1,2} \tag{5}$$

Finding $y_1(t)$ and $y_2(t)$, when $R^2C^2 - 4LC \neq 0$: In the event that $R^2C^2 - 4LC \neq 0$, that is $R \neq 2\sqrt{\frac{L}{C}}$, r_1 and r_2 are different and we may take $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ so that $C_1 y_1(t) + C_2 y_2(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. When $R^2C^2 - 4LC < 0$, that is $R < 2\sqrt{\frac{L}{C}}$, r_1 and r_2 are the two complex numbers $-\rho \pm i\nu$, where $\rho = \frac{R}{2L}$ and $\nu = \frac{\sqrt{4LC-R^2C^2}}{2LC}$. We can rewrite the complementary solution $C_1 e^{r_1 t} + C_2 e^{r_2 t}$ in terms of real valued functions by using that $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$:

$$\begin{aligned} C_1 e^{r_1 t} + C_2 e^{r_2 t} &= e^{-\rho t}\big[C_1 e^{i\nu t} + C_2 e^{-i\nu t}\big] \\ &= e^{-\rho t}\big[C_1\big\{\cos(\nu t) + i\sin(\nu t)\big\} + C_2\big\{\cos(\nu t) - i\sin(\nu t)\big\}\big] \\ &= e^{-\rho t}\big[D_1\cos(\nu t) + D_2\sin(\nu t)\big] \end{aligned}$$

where[(1)] $D_1 = C_1 + C_2,\ D_2 = i(C_1 - C_2)$. So we may also take $y_1(t) = e^{-\rho t}\cos(\nu t)$, $y_2(t) = e^{-\rho t}\cos(\nu t)$ in the complementary solution. There is yet a third useful way to

(1) Don't make the mistake of thinking that C_1 and C_2 have to be real numbers, forcing D_2 to be pure imaginary. In most applications, D_1 and D_2 will be pure real and C_1 and C_2 will be complex.

write the complementary solution. Think of (D_1, D_2) as a point in the xy–plane. Call the polar coordinates of that point R and θ so that $D_1 = R\cos\theta$ and $D_2 = R\sin\theta$. Then, using the trig identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$, with $A = \nu t$ and $B = -\theta$,

$$\begin{aligned} e^{-\rho t}\big[D_1\cos(\nu t) + D_2\sin(\nu t)\big] &= e^{-\rho t}\big[R\cos(\nu t)\cos\theta + R\sin(\nu t)\sin\theta\big] \\ &= Re^{-\rho t}\cos(\nu t - \theta) \end{aligned}$$

We have, in effect, replaced the two arbitrary constants D_1 and D_2, whose values would normally be determined by initial conditions, by two other arbitrary constants, R and θ, whose values would also normally be determined by initial conditions,

Finding $y_1(t)$ and $y_2(t)$, when $R^2C^2 - 4LC = 0$: In the event that $R = 2\sqrt{\frac{L}{C}}$, $r_1 = r_2$. Then we may take $y_1 = e^{r_1 t}$, but $e^{r_2 t} = e^{r_1 t}$ is certainly not a second independent solution. So we still need to find y_2. Here is a trick (called reduction of order) for finding the other solutions: look for solutions of the form $v(t)e^{-r_1 t}$. Here $e^{-r_1 t}$ is the solution we have already found and $v(t)$ is to be determined. To save writing, set $\rho = \frac{R}{2L}$ so that $r_1 = r_2 = \rho$. To save writing also divide $(4_{\rm h})$ by LC and substitute that $\frac{R}{L} = 2\rho$ and $\frac{1}{LC} = \frac{R^2}{4L^2} = \rho^2$. (Recall that we are assuming that $R^2 = \frac{4L}{C}$.) So $(4_{\rm h})$ is equivalent to

$$y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) = 0$$

Sub in

$$\begin{aligned} y_h(t) &= \quad v(t)e^{-\rho t} \\ y_h'(t) &= -\rho\ v(t)e^{-\rho t} + \quad v'(t)e^{-\rho t} \\ y_h''(t) &= \ \rho^2 v(t)e^{-\rho t} - 2\rho v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \end{aligned}$$

Thus when $y_h(t) = v(t)e^{-\rho t}$,

$$\begin{aligned} y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) &= \big[\rho^2 - 2\rho^2 + \rho^2\big]v(t)e^{-\rho t} + \big[-2\rho + 2\rho\big]v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \\ &= v''(t)e^{-\rho t} \end{aligned}$$

Thus $v(t)e^{-\rho t}$ is a solution of $(4_{\rm h})$ whenever the function $v''(t) = 0$ for all t. But, for any values of the constants C_1 and C_2, $v(t) = C_1 + C_2 t$ has vanishing second derivative so $\big(C_1 + C_2 t\big)e^{-\rho t} = \big(C_1 + C_2 t\big)e^{-r_1 t}$ solves $(4_{\rm h})$. This is of the form $C_1 y_1(t) + C_2 y_2(t)$ with $y_1(t) = e^{-r_1 t}$, the solution that we found first, and $y_2(t) = te^{-r_1 t}$, a second independent solution. So we may take $y_2(t) = te^{r_1 t}$.

Finding $y_p(t)$: The best way to find y_p is to guess it. We guess that the circuit responds to an oscillating input voltage with an output voltage that oscillates at the same frequency. So we try $y_p(t) = \mathcal{A}\sin(\omega t - \varphi)$ with the amplitude $\mathcal{A}$ and phase φ to be determined. For

$y_p(t)$ to be a solution, we need

$$\begin{aligned} LCy_p''(t) + RCy_p'(t) + y_p(t) &= E_0 \sin(\omega t) \qquad (4_p) \\ -LC\omega^2\mathcal{A}\sin(\omega t-\varphi) + RC\omega\mathcal{A}\cos(\omega t-\varphi) + \mathcal{A}\sin(\omega t-\varphi) &= E_0\sin(\omega t) \\ &= E_0\sin(\omega t-\varphi+\varphi) \end{aligned}$$

and hence, applying $\sin(A+B) = \sin A\cos B + \cos A\sin B$ with $A = \omega t - \varphi$ and $B = \varphi$,

$$\big(1-LC\omega^2\big)\mathcal{A}\sin(\omega t-\varphi)+RC\omega\mathcal{A}\cos(\omega t-\varphi) = E_0\cos(\varphi)\sin(\omega t-\varphi)+E_0\sin(\varphi)\cos(\omega t-\varphi)$$

Matching coefficients of $\sin(\omega t-\varphi)$ and $\cos(\omega t-\varphi)$ on the left and right hand sides gives

$$\big(1-LC\omega^2\big)\mathcal{A} = E_0\cos(\varphi) \qquad (6)$$

$$RC\omega\mathcal{A} = E_0\sin(\varphi) \qquad (7)$$

It is now easy to solve for $\mathcal{A}$ and φ

$$\frac{(7)}{(6)} \implies \tan(\varphi) = \frac{RC\omega}{1-LC\omega^2} \qquad \implies \varphi = \tan^{-1}\frac{RC\omega}{1-LC\omega^2}$$

$$\sqrt{(6)^2+(7)^2} \implies \sqrt{\big(1-LC\omega^2\big)^2+R^2C^2\omega^2}\,\mathcal{A} = E_0 \implies \mathcal{A} = \frac{E_0}{\sqrt{(1-LC\omega^2)^2+R^2C^2\omega^2}} \qquad (8)$$