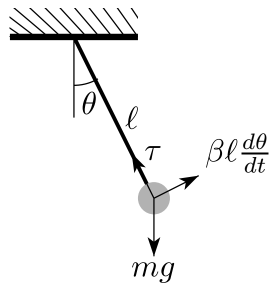
The Pendulum

Model a pendulum by a mass m that is connected to a hinge by an idealized rod that is massless and of fixed length ℓ. Denote by θ the angle between the rod and vertical.



The forces acting on the mass are

- gravity, which has magnitude mg and direction $(0,-1)$,
- tension in the rod, whose magnitude, τ, automatically adjusts itself so that the distance between the mass and the hinge is fixed at ℓ and whose direction, $(-\sin\theta,\cos\theta)$, is always parallel to the rod and
- possibly some frictional forces, like friction in the hinge and air resistance. We shall assume that the total frictional force has magnitude proportional to the speed of the mass and has direction opposite to the direction of motion of the mass.

If we use a coordinate system centered on the hinge, the (x,y) coordinates of the mass are $\ell(\sin\theta,-\cos\theta)$. Hence its velocity vector is $\ell(\cos\theta,\sin\theta)\frac{d\theta}{dt}$ and the total frictional force is $-\beta\ell(\cos\theta,\sin\theta)\frac{d\theta}{dt}$, for some constant β. The acceleration vector of the mass is $\ell(\cos\theta,\sin\theta)\frac{d^2\theta}{dt^2}+\ell(-\sin\theta,\cos\theta)\big(\frac{d\theta}{dt}\big)^2$ so that Newton's law of motion now tells us

$$\begin{aligned} m\ell(\cos\theta,\sin\theta)\tfrac{d^2\theta}{dt^2}&+m\ell(-\sin\theta,\cos\theta)\big(\tfrac{d\theta}{dt}\big)^2\\ &=mg(0,-1)+\tau(-\sin\theta,\cos\theta)-\beta\ell(\cos\theta,\sin\theta)\tfrac{d\theta}{dt}\end{aligned}$$

Dotting this with $(\cos\theta,\sin\theta)$ so as to extract the components parallel to the direction of motion of the mass gives $m\ell\frac{d^2\theta}{dt^2}=-mg\sin\theta-\beta\ell\frac{d\theta}{dt}$ or

$$\boxed{\frac{d^2\theta}{dt^2}+\frac{\beta}{m}\frac{d\theta}{dt}+\frac{g}{\ell}\sin\theta=0}$$

which is the equation of the nonlinear pendulum. If the amplitude of oscillation is small enough that we may approximate $\sin\theta$ by θ we get the equation of the linear pendulum which is

$$\boxed{\frac{d^2\theta}{dt^2}+\frac{\beta}{m}\frac{d\theta}{dt}+\frac{g}{\ell}\theta=0}$$

These equations may be reformulated as systems of first order ordinary differential equation, that is as equations for the flow lines of a vector field, by the simple expedient of defining

$$x(t) = \theta(t) \qquad y(t) = \theta'(t)$$

Then, for the full, nonlinear, equation $\frac{d^2\theta}{dt^2} + \frac{\beta}{m}\frac{d\theta}{dt} + \frac{g}{\ell}\sin\theta = 0$

$$\begin{aligned} x'(t) &= \theta'(t) \ = y(t) \\ y'(t) &= \theta''(t) = -\tfrac{g}{\ell}\sin x(t) - \tfrac{\beta}{m}y(t) \end{aligned}$$

The solutions of this first order system of ordinary differential equations are flow lines for the vector field

$$\mathbf{V}\big((x,y)\big) = \big(y\,,\ -\tfrac{g}{\ell}\sin x - \tfrac{\beta}{m}y\big)$$