Poisson's Equation

In these notes we shall find a formula for the solution of Poisson's equation

$$\nabla^2\varphi = 4\pi\rho$$

Here ρ is a given (smooth) function and φ is the unknown function. In electrostatics, ρ is the charge density and φ is the electric potential. The main step in finding this formula will be to consider an

arbitrary (smooth) function φ and an
arbitrary (smooth) region V in $\mathbb{R}^3$ and an
arbitrary point $\mathbf{r}_0$ in the interior of V

and to find a formula which expresses $\varphi(\mathbf{r}_0)$ in terms of

$\nabla^2\varphi(\mathbf{r})$, with $\mathbf{r}$ running over V and
$\nabla\varphi(\mathbf{r})$ and $\varphi(\mathbf{r})$, with $\mathbf{r}$ running only over ∂V.

This formula is

$$\varphi(\mathbf{r}_0) = -\tfrac{1}{4\pi}\Big\{ \iiint_V \tfrac{\nabla^2\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\ d^3\mathbf{r} - \iint_{\partial V} \varphi(\mathbf{r})\tfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}\cdot\hat{\mathbf{n}}\ dS - \iint_{\partial V} \tfrac{\nabla\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\cdot\hat{\mathbf{n}}\ dS\Big\} \qquad (V)$$

When we take the limit as V expands to fill all of $\mathbb{R}^3$ (assuming that φ and $\nabla\varphi$ go to zero sufficiently quickly at ∞), we will end up with the formula

$$\varphi(\mathbf{r}_0) = -\tfrac{1}{4\pi}\iiint_V \tfrac{\nabla^2\varphi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|}\ d^3\mathbf{r}$$

that expresses φ evaluated at an arbitrary point, $\mathbf{r}_0$, of $\mathbb{R}^3$ in terms of $\nabla^2\varphi(\mathbf{r})$, with $\mathbf{r}$ running over $\mathbb{R}^3$, which is exactly what we want.

Let

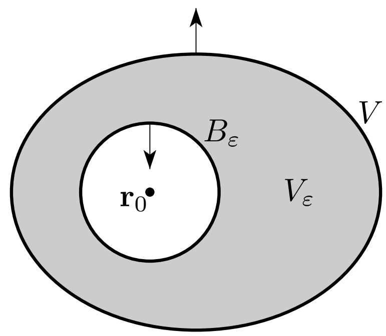
$$\mathbf{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$$
$$\mathbf{r}_0 = x_0\,\hat{\imath} + y_0\,\hat{\jmath} + z_0\,\hat{\mathbf{k}}$$

We shall exploit several properties of the function $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$. The first two properties are

$$\nabla\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\tfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$$
$$\nabla^2\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|} = -\nabla\cdot\tfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} = 0$$

and are valid for all $\mathbf{r}\neq\mathbf{r}_0$. Verifying these properties are simple two line computations. The other property of $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}$ that we shall use is the following. Let B_ε be the sphere of radius ε centered on $\mathbf{r}_0$. Then, for any continuous function $\psi(\mathbf{r})$,

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\tfrac{\psi(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|^p}\ dS &= \lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\tfrac{\psi(\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^p}\ dS = \lim_{\varepsilon\to0+}\tfrac{\psi(\mathbf{r}_0)}{\varepsilon^p}\iint_{B_\varepsilon} dS = \lim_{\varepsilon\to0+}\tfrac{\psi(\mathbf{r}_0)}{\varepsilon^p}4\pi\varepsilon^2\\ &= \begin{cases}4\pi\psi(\mathbf{r}_0) & \text{if } p=2\\ 0 & \text{if } p<2\end{cases}\end{aligned} \qquad \text{(B)}$$



Here is the derivation of (V). Let V_ε be the part of V outside of B_ε. Note that the boundary ∂V_ε of V_ε consists of two parts — the boundary ∂V of V and the sphere B_ε — and that the unit outward normal to ∂V_ε on B_ε is $-\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}$. By the divergence theorem

$$\begin{aligned}\iiint_{V_\varepsilon} \boldsymbol{\nabla}\cdot\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big)dV &= \iint_{\partial V}\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big)\cdot\hat{\mathbf{n}}\ dS \\ &\quad+\iint_{B_\varepsilon}\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big)\cdot\big(-\tfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}\big)dS\end{aligned}\tag{M}$$

Subbing in $\boldsymbol{\nabla}\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=-\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$ and applying (B)

$$\begin{aligned}\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big)\cdot\big(-\tfrac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}\big)dS &= -\lim_{\varepsilon\to0+}\iint_{B_\varepsilon}\big(\boldsymbol{\nabla}\varphi\cdot(\mathbf{r}-\mathbf{r}_0)+\varphi\big)\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|^2}dS \\ &= -4\pi\Big[\boldsymbol{\nabla}\varphi\cdot(\mathbf{r}-\mathbf{r}_0)+\varphi\Big]_{\mathbf{r}=\mathbf{r}_0} \\ &= -4\pi\varphi(\mathbf{r}_0)\end{aligned}\tag{R}$$

Applying $\boldsymbol{\nabla}\cdot\big(f\mathbf{F}\big)=\boldsymbol{\nabla}f\cdot\mathbf{F}+f\boldsymbol{\nabla}\cdot\mathbf{F}$, twice, we see that the integrand of the left hand side is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big) &= \boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\cdot\boldsymbol{\nabla}\varphi+\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}^2\varphi-\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}-\varphi\boldsymbol{\nabla}^2\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|} \\ &= \tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}^2\varphi\end{aligned}\tag{L}$$

since $\boldsymbol{\nabla}^2\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=0$ on V_ε. So applying $\lim_{\varepsilon\to0+}$ to (M) and applying (L) and (R) gives

$$\iiint_V \tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}^2\varphi\ dV=\iint_{\partial V}\big(\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\boldsymbol{\nabla}\varphi-\varphi\boldsymbol{\nabla}\tfrac{1}{|\mathbf{r}-\mathbf{r}_0|}\big)\cdot\hat{\mathbf{n}}\ dS-4\pi\varphi(\mathbf{r}_0)$$

which is exactly equation (V).