Stokes' Theorem

The statement

Let

- S be a smooth oriented surface (i.e. a unit normal $\hat{\mathbf{n}}$ has been chosen at each point of S and this choice depends continuously on the point)
- the boundary, ∂S, of the surface S also be smooth and be oriented consistently with $\hat{\mathbf{n}}$ in the sense that
  - if you stand on S with the vector from your feet to your head having direction $\hat{\mathbf{n}}$ and
  - if you walk along ∂S in the direction of the arrow on ∂S,
  - then S is on your left hand side.
- $\mathbf{F}$ be a vector field that has continuous first partial derivatives at every point of S

Then

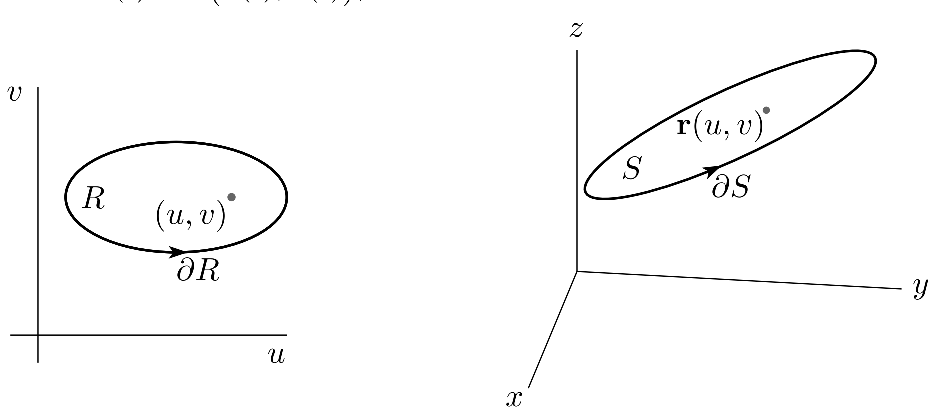
$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

The proof Both integrals involve F_1 terms and F_2 terms and F_3 terms. We shall show that the F_1 terms in the two integrals agree. In other words, we shall assume that $\mathbf{F} = F_1 \hat{\boldsymbol{\imath}}$. The proofs that the F_2 and F_3 terms also agree are similar.

Pick a parametrization of S with

$$S = \left\{ \mathbf{r}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \;\middle|\; (u,v) \text{ in } R \subset \mathbb{R}^2 \right\}$$

and with $\mathbf{r}(u,v)$ orientation preserving in the sense that $\hat{\mathbf{n}}\, dS = +\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv$. Pick a parametrization of the curve, ∂R, bounding R as $\big(u(t), v(t)\big)$, $a \le t \le b$, in such a way that the velocity vector $\big(u'(t), v'(t)\big)$ is never zero, and when you walk along ∂R in the direction of increasing t, then R is on your left. Then the curve ∂S bounding S can be parametrized as $\mathbf{R}(t) = \mathbf{r}\big(u(t), v(t)\big)$, $a \le t \le b$.



The orientation of $\mathbf{R}(t)$ I claim that the direction of increasing t for the parametrization $\mathbf{R}(t)$ is the direction of the arrow on ∂S. For simplicity, suppose that ∂R is connected.

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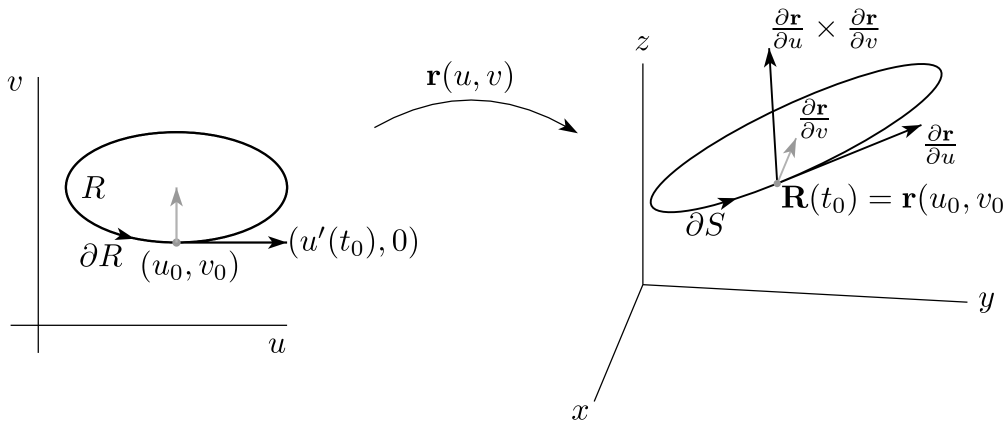
By continuity, it suffices to check the orientation at a single point. Find a point (u_0, v_0) on ∂R where the forward pointing tangent vector is a positive multiple of $\hat{\boldsymbol{\imath}}$. The arrow on ∂R in the figure below is at such a point. Suppose that $t = t_0$ at this point — in other words, suppose that $(u_0, v_0) = \big(u(t_0), v(t_0)\big)$. Because the forward pointing tangent vector to ∂R at (u_0, v_0) is a positive multiple of $\hat{\boldsymbol{\imath}}$, we have $u'(t_0) > 0$ and $v'(t_0) = 0$. The tangent vector to ∂S at $\mathbf{R}(t_0) = \mathbf{r}\big(u_0, v_0\big)$, pointing in the direction of increasing t, is $\mathbf{R}'(t_0) = \frac{d}{dt}\mathbf{r}\big(u(t), v(t)\big)\big|_{t=t_0} = u'(t_0)\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) + v'(t_0)\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) = u'(t_0)\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$ and so is a positive multiple of $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$.

If we now walk along a path in the uv–plane which starts at (u_0, v_0), holds u fixed at u_0 and increases v, we move into the interior of R starting at (u_0, v_0). Correspondingly, if we walk along a path, $\mathbf{r}(u_0, v)$, in $\mathbb{R}^3$ with v starting at v_0 and increasing, we move into the interior of S. The forward tangent to this new path, $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$, points from $\mathbf{r}(u_0, v_0)$ into the interior of S.



Now imagine that you are walking along ∂S in the direction of increasing t. At time t_0 you are at $\mathbf{R}(t_0)$. You point your right arm straight ahead of you. So it is pointing in the direction $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)$. You point your left arm straight out sideways into the interior of S. It is pointing in the direction $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$. If the direction of increasing t is the same as the forward direction of the orientation of ∂S, then the vector from our feet to our head, which is $\frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) \times \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)$, should be pointing in the same direction as $\hat{\mathbf{n}}$. Since $\hat{\mathbf{n}}\, dS = +\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv$, it is.

The surface integral Since $F = F_1 \hat{\boldsymbol{\imath}}$ and $\hat{\mathbf{n}}\, dS = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv$,

$$\begin{aligned}\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_R \big(0, \tfrac{\partial F_1}{\partial z}, -\tfrac{\partial F_1}{\partial y}\big) \cdot \tfrac{\partial \mathbf{r}}{\partial u} \times \tfrac{\partial \mathbf{r}}{\partial v}\, du\, dv \\ &= \iint_R \Big\{ \tfrac{\partial F_1}{\partial z}\big(\tfrac{\partial z}{\partial u}\tfrac{\partial x}{\partial v} - \tfrac{\partial x}{\partial u}\tfrac{\partial z}{\partial v}\big) - \tfrac{\partial F_1}{\partial y}\big(\tfrac{\partial x}{\partial u}\tfrac{\partial y}{\partial v} - \tfrac{\partial y}{\partial u}\tfrac{\partial x}{\partial v}\big)\Big\}\, du\, dv\end{aligned}$$

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The line integral

$$\begin{aligned}\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}\Big(\mathbf{r}\big(u(t),v(t)\big)\Big)\cdot \tfrac{d}{dt}\mathbf{r}\big(u(t),v(t)\big)\ dt\\ &= \int_a^b \mathbf{F}\Big(\mathbf{r}\big(u(t),v(t)\big)\Big)\cdot \Big[\tfrac{\partial \mathbf{r}}{\partial u}\big(u(t),v(t)\big)\tfrac{du}{dt}(t)+\tfrac{\partial \mathbf{r}}{\partial v}\tfrac{dv}{dt}\Big]\ dt\end{aligned}$$

This agrees exactly with the line integral

$$\oint_{\partial R} M(u,v)\ du + N(u,v)\ dv = \int_a^b \Big[M\big(u(t),v(t)\big)\ \tfrac{du}{dt}(t) + N\big(u(t),v(t)\big)\ \tfrac{dv}{dt}(t)\Big]\ dt$$

around ∂R, if we choose

$$\begin{aligned}M(u,v) &= \mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot\tfrac{\partial \mathbf{r}}{\partial u}(u,v)\\ &= F_1\big(x(u,v),y(u,v),z(u,v)\big)\tfrac{\partial x}{\partial u}(u,v)\\ N(u,v) &= \mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot\tfrac{\partial \mathbf{r}}{\partial v}(u,v)\\ &= F_1\big(x(u,v),y(u,v),z(u,v)\big)\tfrac{\partial x}{\partial v}(u,v)\end{aligned}$$

By Green's Theorem, we have

$$\begin{aligned}&\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r} = \oint_{\partial R} M(u,v)\ du + N(u,v)\ dv\\ &= \iint_R \Big\{\tfrac{\partial N}{\partial u} - \tfrac{\partial M}{\partial v}\Big\}\ dudv\\ &= \iint_R \Big\{\big(\tfrac{\partial F_1}{\partial x}\tfrac{\partial x}{\partial u} + \tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial u} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial u}\big)\tfrac{\partial x}{\partial v} + F_1\tfrac{\partial^2 x}{\partial u\partial v}\\ &\qquad -\big(\tfrac{\partial F_1}{\partial x}\tfrac{\partial x}{\partial v} + \tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial v} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial v}\big)\tfrac{\partial x}{\partial u} - F_1\tfrac{\partial^2 x}{\partial v\partial u}\Big\}\ dudv\\ &= \iint_R \Big\{\big(\tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial u} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial u}\big)\tfrac{\partial x}{\partial v} - \big(\tfrac{\partial F_1}{\partial y}\tfrac{\partial y}{\partial v} + \tfrac{\partial F_1}{\partial z}\tfrac{\partial z}{\partial v}\big)\tfrac{\partial x}{\partial u}\Big\}\,du\,dv\\ &\quad = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS\end{aligned}$$

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