Example of the Use of Stokes' Theorem

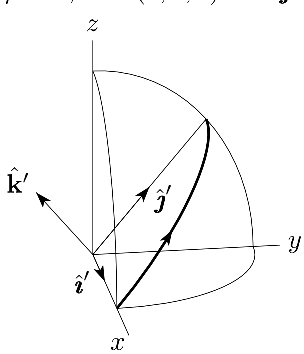
In these notes we compute, in three different ways, $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (z-y)\,\hat{\boldsymbol{\imath}} - (x+z)\,\hat{\boldsymbol{\jmath}} - (x+y)\,\hat{\mathbf{k}}$ and C is the curve $x^2+y^2+z^2=4$, $z=y$ oriented counterclockwise when viewed from above.

Direct Computation

In this first computation, we parametrize the curve C and compute $\oint_C \vec{F}\cdot d\vec{r}$ directly. The plane $z=y$ passes through the origin, which is the centre of the sphere $x^2+y^2+z^2=4$. So C is a circle which, like the sphere, has radius 2 and centre $(0,0,0)$. We use a parametrization of the form

$$\vec{r}(t) = \vec{c} + \rho\cos t\,\hat{\boldsymbol{\imath}}' + \rho\sin t\,\hat{\boldsymbol{\jmath}}' \qquad 0\le t\le 2\pi$$

where $\vec{c}=(0,0,0)$ is the centre of C, $\rho=2$ is the radius of C and $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ are two vectors that (a) are unit vectors, (b) are parallel to the plane $z=y$ and (c) are mutually perpendicular. The point $(2,0,0)$ satisfies both $x^2+y^2+z^2=4$ and $z=y$ and so is on C. We may choose $\hat{\boldsymbol{\imath}}'$ to be the unit vector in the direction from the centre $(0,0,0)$ of the circle towards $(2,0,0)$. Namely $\hat{\boldsymbol{\imath}}'=(1,0,0)$. Since the plane of the circle is $z-y=0$, the vector $\vec{\nabla}(z-y)=(0,-1,1)$ is perpendicular to the plane of C. So $\hat{\mathbf{k}}'=\frac{1}{\sqrt{2}}(0,-1,1)$ is a unit vector normal to $z=y$. Then $\hat{\boldsymbol{\jmath}}'=\hat{\mathbf{k}}'\times\hat{\boldsymbol{\imath}}'=\frac{1}{\sqrt{2}}(0,-1,1)\times(1,0,0)=\frac{1}{\sqrt{2}}(0,1,1)$ is a unit vector that is perpendicular to $\hat{\boldsymbol{\imath}}'$. Since $\hat{\boldsymbol{\jmath}}'$ is also perpendicular to $\hat{\mathbf{k}}'$, it is parallel to $z=y$. Subbing in $\vec{c}=(0,0,0)$, $\rho=2$, $\hat{\boldsymbol{\imath}}'=(1,0,0)$ and $\hat{\boldsymbol{\jmath}}'=\frac{1}{\sqrt{2}}(0,1,1)$ gives



$$\vec{r}(t) = 2\cos t\,(1,0,0) + 2\sin t\,\tfrac{1}{\sqrt{2}}(0,1,1) = 2\big(\cos t, \tfrac{\sin t}{\sqrt{2}}, \tfrac{\sin t}{\sqrt{2}}\big) \qquad 0\le t\le 2\pi$$

To check that this parametrization is correct, note that $x=2\cos t$, $y=\sqrt{2}\sin t$, $z=\sqrt{2}\sin t$ satisfies both $x^2+y^2+z^2=4$ and $z=y$. At $t=0$, $\vec{r}(0)=(2,0,0)$. As t increases, $z(t)=\sqrt{2}$ increases and $\vec{r}(t)$ moves upwards towards $\vec{r}\big(\frac{\pi}{2}\big)=(0,\sqrt{2},\sqrt{2})$. This is the desired counterclockwise direction. Now that we have a parametrization, we can set up the integral.

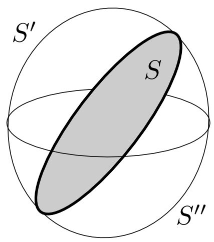
$$\begin{aligned}
\vec{r}(t) &= \big(2\cos t, \sqrt{2}\sin t, \sqrt{2}\sin t\big)\\
\vec{r}\,'(t) &= \big(-2\sin t, \sqrt{2}\cos t, \sqrt{2}\cos t\big)\\
\vec{F}\big(\vec{r}(t)\big) &= \big(z(t)-y(t), -x(t)-z(t), -x(t)-y(t)\big)\\
&= \big(\sqrt{2}\sin t-\sqrt{2}\sin t, -2\cos t-\sqrt{2}\sin t, -2\cos t-\sqrt{2}\sin t\big)\\
&= -\big(0, 2\cos t+\sqrt{2}\sin t, 2\cos t+\sqrt{2}\sin t\big)\\
\vec{F}\big(\vec{r}(t)\big)\cdot\vec{r}\,'(t) &= -\big[4\sqrt{2}\cos^2 t + 4\cos t\sin t\big] = -\big[2\sqrt{2}\cos(2t)+2\sqrt{2}+2\sin(2t)\big]\\
\oint_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \vec{F}\big(\vec{r}(t)\big)\cdot\vec{r}\,'(t)\ dt\\
&= \int_0^{2\pi} -\big[2\sqrt{2}\cos(2t)+2\sqrt{2}+2\sin(2t)\big]\ dt = -\big[\sqrt{2}\sin(2t)+2\sqrt{2}t-\cos(2t)\big]_0^{2\pi} = \boxed{-4\sqrt{2}\pi}
\end{aligned}$$

Stokes' Theorem

To apply Stokes' theorem we need to express C as the boundary ∂S of a surface S. As

$$C = \left\{\ (x,y,z)\ \middle|\ x^2+y^2+z^2=4,\ z=y\ \right\}$$

is a closed curve, this is possible. In fact there are many possible choices of S with $\partial S = C$. Three possible S's are



$$\begin{aligned} S &= \left\{\ (x,y,z)\ \middle|\ x^2+y^2+z^2\le 4,\ z=y\ \right\} \\ S' &= \left\{\ (x,y,z)\ \middle|\ x^2+y^2+z^2= 4,\ z\ge y\ \right\} \\ S'' &= \left\{\ (x,y,z)\ \middle|\ x^2+y^2+z^2= 4,\ z\le y\ \right\} \end{aligned}$$

The first of these, which is part of a plane, is likely to lead to simpler computations than the last two, which are parts of a sphere. So we choose to use it.

In preparation for application of Stokes' theorem, we compute $\vec{\nabla}\times\vec{F}$ and $\hat{\mathbf{n}}\,dS$. For the latter, we apply the formula $\hat{\mathbf{n}}\,dS = \pm(-f_x,-f_y,1)\,dxdy$ to the surface $z=f(x,y)=y$. We use the $+$ sign to give the normal a positive $\hat{\mathbf{k}}$ component.

$$\begin{aligned} \vec{\nabla}\times\vec{F} &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & -x-z & -x-y \end{bmatrix} = \hat{\imath}\big(-1-(-1)\big) - \hat{\jmath}\big(-1-1\big) + \hat{\mathbf{k}}\big(-1-(-1)\big) = 2\,\hat{\jmath} \\ \hat{\mathbf{n}}\,dS &= (0,-1,1)\,dxdy \\ \vec{\nabla}\times\vec{F}\cdot\hat{\mathbf{n}}\,dS &= (0,2,0)\cdot(0,-1,1)\,dxdy = -2\,dxdy \end{aligned}$$

The integration variables are x and y and, by definition, the domain of integration is

$$R = \left\{\ (x,y)\ \middle|\ (x,y,z) \text{ is in } S \text{ for some } z\ \right\}$$

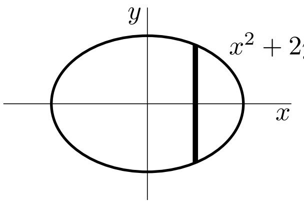
To determine precisely what this domain of integration is, we observe that since $z=y$ on S

$$S = \left\{\ (x,y,z)\ \middle|\ x^2+2y^2\le 4,\ z=y\ \right\} \implies R = \left\{\ (x,y)\ \middle|\ x^2+2y^2\le 4\ \right\}$$

So the domain of integration is an ellipse with semimajor axis $a=2$, semiminor axis $b=\sqrt{2}$ and area $\pi ab = 2\sqrt{2}\pi$ and

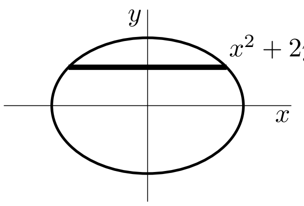
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{\mathbf{n}}\,dS = \iint_R (-2)\,dxdy = -2\ \text{Area}\,(R) = \boxed{-4\sqrt{2}\pi}$$

Remark (Limits of integration) If the integrand were more complicated, we would have to evaluate the integral over R by expressing it as an iterated integrals with the correct limits of integration. First suppose that we slice up R using thin vertical slices. On each such slice, x is essentially constant and y runs from $-\sqrt{(4-x^2)/2}$ to $\sqrt{(4-x^2)/2}$. The leftmost such slice would have $x=-2$ and the rightmost such slice would have $x=2$. So the correct limits with this slicing are



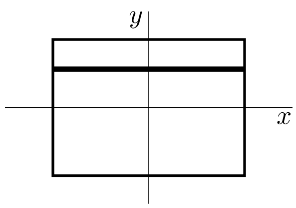
$$\iint_R f(x,y)\,dxdy = \int_{-2}^{2} dx \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} dy\ f(x,y)$$

If, instead, we slice up R using thin horizontal slices, then, on each such slice, y is essentially constant and x runs from $-\sqrt{4-2y^2}$ to $\sqrt{4-2y^2}$. The bottom such slice would have $y = -\sqrt{2}$ and the top such slice would have $y = \sqrt{2}$. So the correct limits with this slicing are



$$\iint_R f(x,y)\,dxdy = \int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dx\ f(x,y)$$

Note that the integral with limits



$$\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-2}^{2} dx\ f(x,y)$$

corresponds to a slicing with x running from -2 to 2 on **every** slice. This corresponds to a rectangular domain of integration.

Stokes' Theorem, Again

Since the integrand is just a constant and S is so simple, we can evaluate the integral $\iint_S \vec{\nabla} \times \vec{F} \cdot \hat{\mathbf{n}}\,dS$ without ever determining dS explicitly and without ever setting up any limits of integration. We already know that $\vec{\nabla} \times \vec{F} = 2\,\hat{\boldsymbol{\jmath}}$. Since S is the level surface $z - y = 0$, the gradient $\vec{\nabla}(z-y) = -\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$ is normal to S. So $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$ and

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{\mathbf{n}}\,dS = \iint_S (2\hat{\boldsymbol{\jmath}}) \cdot \tfrac{1}{\sqrt{2}}(-\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\,dS = \iint_S -\sqrt{2}\,dS = -\sqrt{2}\ \text{Area}\,(S)$$

As S is a circle of radius 2, $\boxed{\oint_C \vec{F} \cdot d\vec{r} = -4\sqrt{2}\pi}$, yet again.